Today’s Outline

- Flow Networks & Maximum Flow
  - Flow networks
  - Residual networks, augmenting paths, cuts
  - Max-flow min-cut theorem
  - Ford-Fulkerson method
Flow Networks: Intro

- A digraph can be viewed as a flow network
- Can answer questions about material flows from a source (produce) to a sink (consume)
- Produce and consume at same rate
- Flow of material at a point in the system is the rate at which material moves
Flow Networks: Intro

- Examples
  - Liquids flowing through pipes
  - Parts through assembly lines
  - Current through electrical wires
  - Information through communication networks
Flow Networks: Intro

- **Examples**
  - Liquids flowing through pipes
  - Parts through assembly lines
  - Current through electrical wires
  - Information through communication networks

- **An edge ~ a conduit for material**
  - Has a stated *capacity* (e.g., 200 gallons/hour of liquid, 20 amperes of current)
Flow Networks: Intro

- Vertices are conduit junctions
  - Material flows, without collecting in vertices
  - (except source and sink)
- **Flow conservation** at vertices
  - Rate at which material enters a vertex = rate at which it leaves the vertex
  - (same as Kirchchoff’s Current Law)
Flow Networks: Intro

- **Maximum flow problem** (in simple form)

  - What is the greatest rate at which material can be sent (shipped) from the source to the sink without violating any capacity constraints?

- Can be solved by efficient algorithms
Definitions etc.

- **Flow network** $G=(V,E)$ is a directed graph
  - each edge $(u,v)$ in $E$ has a nonnegative capacity $c(u,v) \geq 0$
  - if $(u,v)$ is not in $E$ then we assume $c(u,v)=0$
  - two special vertices, source, $s$, and sink, $t$
  - assume every vertex is on some $s$-$t$ path

- Example: CLRS, Fig 26.1(a)
Fig 26.1 (a) in CLRS
Let $G=(V,E)$ be a flow network with capacity function $c$, source $s$ and sink $t$

A flow in $G$ is a real-valued function $f: V \times V \rightarrow \mathbb{R}$ that satisfies 3 properties:

1. for all $u, v$ in $V$, $f(u, v) \leq c(u, v)$
2. for all $u, v$ in $V$, $f(u, v) = -f(v, u)$
3. for all $u$ in $V-\{s,t\}$

$$\sum_{v \in V} f(u, v) = 0$$

capacity constraint
skew symmetry
flow conservation
Definitions etc.

- The net flow from vertex u to vertex v is the quantity $f(u,v)$ which can be +ve or –ve.
- The value of a flow $f$ is $(total\ net\ flow\ out\ of\ s)$
  \[ |f| = \sum_{v \in V} f(s, v) \]
- In the maximum-flow problem, given $G$ with $s$ and $t$, we wish to find a flow of maximum value from $s$ to $t$.
Network Flow: An Example

Fig 26.1 (b) in CLRS
A flow with value $|f| = 19$
Multiple Sources, Sinks?

- What if there are > 1 sources and sinks?
  - E.g., company with \( m \) factories \( n \) warehouses

- Can reduce to an ordinary maximum-flow problem
  - Can add a \textit{supersource} and a \textit{supersink}
  - E.g., Fig. 26.2, p. 648 in CLRS
  - Can prove the two problems are equivalent
Multiple Sources, Sinks?

Fig 26.2 in CLRS
Ford-Fulkerson Method

- Solves the maximum-flow problem
  - Involves several implementations/algorithms
- Depends on 3 (broad) ideas
  - Residual networks
  - Augmenting paths
  - Cuts
- And the max-flow min-cut theorem
Ford-Fulkerson Method

- Iterative method
  - Initialize \( f(u,v) = 0 \) for all \( u, v \) in \( V \)
  - At each iteration, increase the flow by finding an augmenting path
    - (an \( s-t \) path along which we can push more flow)
  - Then augment the flow along this path
  - Repeat until no augmenting path is found
Ford-Fulkerson Method

\[
\text{FORD-FULKERSON-METHOD}(G,s,t) \\
\hspace{1cm} \text{initialize flow } f \text{ to 0} \\
\hspace{1cm} \text{while there exists an augmenting path } p \\
\hspace{1cm} \quad \text{augment flow } f \text{ along } p \\
\hspace{1cm} \text{return } f
\]

- Upon termination, yields a maximum flow
Residual Networks

Given a flow network and a flow, the residual network consists of edges that can admit more net flow.

More formally:

- Suppose flow network $G=(V,E)$, $s$, $t$ are given.
- Let $f$ be a flow in $G$ and $u$, $v$ be vertices in $V$.
- *Additional* net flow we can push from $u$ to $v$ before exceeding capacity $c(u,v)$ is the *residual capacity* of $(u,v)$ given by:

$$c_f(u,v) = c(u,v) - f(u,v)$$
Residual Networks

- **Example**
  - If $c(u,v) = 16$ and $f(u,v) = 11$ then we can ship $c_f(u,v) = 5$ more units of flow.

- **When the net flow is $-$ve, $c_f(u,v) > c(u,v)$**
  - Example: If $c(u,v) = 16$ and $f(u,v) = -4$ then $c_f(u,v) = 20$.
  - This means: push 4 units $u \rightarrow v$ to cancel the 4 units net flow from $v \rightarrow u$, then push 16 more.
Residual Networks

- Given a flow network $G=(V,E)$ and a flow $f$ the residual network of $G$ induced by $f$ is $G_f=(V,E_f)$ where

  $$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

- Each edge of the residual network (residual edge) can admit a positive net flow

- Example: Fig. 26.3(b) on p. 652 in CLRS
Example Residual Network

Network ➔

Residual Network ➔
Residual Networks

- Important property (Lemma 26.2 in CLRS)
  - Let $G = (V, E)$, $s$, $t$ be a flow network and $f$ be a flow in $G$
  - Let $G_f$ be the residual network induced by $f$ and let $f^*$ be a flow in $G_f$
  - Then the flow sum $f + f^*$ is a flow in $G$ with value $|f + f^*| = |f| + |f^*|$

- Shows how a flow in $G_f$ relates to one in $G$
Augmenting Paths

- Given a flow network $G=(V,E)$ and a flow $f$, an augmenting path $p$ is a simple path from $s$ to $t$ in the residual network $G_f$.
  - Each edge $(u,v)$ on the augmenting path admits some additional positive net flow from $u$ to $v$ without violating the capacity constraint.

- Example
  - Fig. 26.3(b) on p. 652 in CLRS
Example Augmenting Path

Network \rightarrow Residual network with augmenting path shaded

Residual network with augmenting path shaded
Augmenting Paths

- Residual capacity $c_f(p)$ of an augmenting path $p$ is the maximum amount of net flow that we can ship along the edges of $p$

$$c_f(p) = \min \{ c_f(u, v) : (u, v) \text{ is on } p \}$$
Augmenting Paths

- Important properties (Lemma 26.3 and Corollary 26.4 in CLRS)
  - Let $G=(V,E)$ be a flow network, $f$ a flow in $G$
  - If $f_p$ is a flow defined on an augmenting path $p$ of a residual network $G_f$ of $G$
  - If we add $f_p$ to $f$, we get another flow in $G$ whose value is closer to the maximum

- Example:
  - Fig. 26.3(c) on p. 652 in CLRS
Is there an augmenting path? → No
What does it mean? → Max flow attained
“Cuts” of Flow Networks

- A cut $(S,T)$ of flow network $G=(V,E)$ is a partition of $V$ into $S$ and $T=V-S$ such that the source $s$ is in $S$ and the sink $t$ is in $T$.

- If $f$ is a flow, then the net flow across the cut $(S,T)$ is $f(S,T)$ and the capacity of the cut $(S,T)$ is $c(S,T)$. 

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28
“Cuts” of Flow Networks

- If \((S, T)\) is a cut in a flow network then
  - the net flow \(f(S, T)\) across the cut \((S, T)\) is

\[
f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)
\]

- The capacity \(c(S, T)\) across the cut \((S, T)\) is

\[
c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)
\]

What is a minimum cut?
Fig 26.4 in CLRS

Net flow across the cut is $f(S, T) = 19$

Capacity across the cut is $c(S, T) = 26$
Max possible flow through the cut = 12 + 7 + 4 = 23
Network has a capacity of at most 23
This is a minimum cut
“Cuts” of Flow Networks

- Property (Lemma 26.5)
  - Let $f$ be a flow in a flow network $G$ and let $(S,T)$ be a cut of $G$. Then the net flow across $(S,T)$ is $f(S,T) = |f|$.

- Corollaries
  - The value of a flow is the net flow into the sink.
  - The value of any flow $f$ in a flow network $G$ is bounded from above by the capacity of any cut of $G$.
The net flow across any cut is the same and equal to the flow of the network $|f| = 23$. 

Example – Net Flow
Max-flow Min-cut Theorem

- If \( f \) is a flow in a flow network \( G=(V,E) \) with source \( s \) and sink \( t \), then the following are equivalent:
  - \( f \) is a maximum flow in \( G \)
  - The residual network \( G_f \) contains no augmenting paths
  - \( |f| = c(S,T) \) for some cut \( (S,T) \) of \( G \)
**Basic Ford-Fulkerson Alg.**

\[\text{Ford-Fulkerson}(G, s, t)\]

1. \textbf{for} each edge \((u, v) \in E[G]\)
2. \hspace{1em} \textbf{do} \(f[u, v] \leftarrow 0\)
3. \hspace{2em} \hspace{1em} \hspace{1em} \(f[v, u] \leftarrow 0\)
4. \textbf{while} there exists a path \(p\) from \(s\) to \(t\) in the residual network \(G_f\)
5. \hspace{1em} \textbf{do} \(c_f(p) \leftarrow \min \{c_f(u, v) : (u, v) \text{ is in } p\}\)
6. \hspace{2em} \hspace{1em} \textbf{for} each edge \((u, v)\) in \(p\)
7. \hspace{2em} \hspace{3em} \textbf{do} \(f[u, v] \leftarrow f[u, v] + c_f(p)\)
8. \hspace{2em} \hspace{3em} \hspace{1em} \(f[v, u] \leftarrow -f[u, v]\)
Example

Original Network

Flow Network

Resulting Flow = 4

augmenting path
Example

Flow Network

Residual Network

Flow Network

Resulting Flow = 4

augmenting path

Resulting Flow = 11
Example

Flow Network

Resulting Flow = 11

Residual Network

augmenting path

Flow Network

Resulting Flow = 19
Example

Flow Network

Residual Network

Flow Network

Resulting Flow = 19

augmenting path

Resulting Flow = 23
Example

Resulting Flow = 23

No augmenting path:
Maxflow=23

Residual Network
**Analysis**

**Ford-Fulkerson** \((G, s, t)\)

1. for each edge \((u, v) \in E[G]\) 
2. \(\text{do } f[u, v] \leftarrow 0\)
3. \(f[v, u] \leftarrow 0\)
4. while there exists a path \(p\) from \(s\) to \(t\) in the residual network \(G_f\) 
5. \(\text{do } c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}\)
6. for each edge \((u, v)\) in \(p\) 
7. \(\text{do } f[u, v] \leftarrow f[u, v] + c_f(p)\)
8. \(f[v, u] \leftarrow -f[u, v]\)

\(O(E)\)
Analysis

- If capacities are all integer, then each augmenting path raises $|f|$ by $\geq 1$
- If max flow is $f^*$, then need $\leq |f^*|$ iterations
  - Running time is $O(E \cdot |f^*|)$
  - This is not polynomial in input size
  - Depends on $|f^*|$, which is not a function of $|V|$ or $|E|$:
- If capacities are rational, can scale to integers
- If irrational, Ford-Fulkerson method might never terminate!
Additional Material

- On network flows, Ford-Fulkerson method and applications
  - Prof. Kincaid’s slides:
    - Slide set 1, Slide set 2

- Read, explore further for
  - Improvements over Ford-Fulkerson approach
    - Edmonds-Karp algorithm
    - Push-relabel algorithm
Conclusion

We discussed

- Flow networks
- Residual networks, augmenting paths, cuts
- *Max-flow min-cut theorem*
- Ford-Fulkerson method

End of discussion on Graph algorithms

Next time

- Computational geometry
References

- The lecture slides are based on the slides prepared by Prof. Sanath Jayasena for this class in previous years.
- CLRS book, 2e, Part VI: Graph Algorithms
  - Chapter 26: Maximum Flow
- Other references
  - Prof. Kincaid: http://www.math.wm.edu/~rrkinc/
  - Prof. James Elder http://elderlab.yorku.ca/~elder/teaching/cse3101/