Announcement

- **Assignment 2** is due on 18\textsuperscript{th} of November
- Worth 5\%
Today’s Outline

• Flow Networks & Maximum Flow
  – Flow networks
  – Residual networks, augmenting paths, cuts
  – \textit{Max-flow min-cut theorem}
  – Ford-Fulkerson method
Flow Networks: Intro

• A digraph can be viewed as a flow network
• Can answer questions about material flows from a source (produce) to a sink (consume)
• Produce and consume at same rate
• Flow of material at a point in the system is the rate at which material moves
Flow Networks: Intro

• Examples
  – liquids flowing through pipes
  – parts through assembly lines
  – current through electrical wires
  – information through communication networks

• An edge ~ a conduit for material
  – Has a stated \textit{capacity} (e.g., 200 gallons/hour of liquid, 20 amperes of current)
Flow Networks: Intro

• Vertices are conduit junctions
  – Material flows, without collecting in vertices
  – (except source and sink)

• Flow conservation at vertices
  – Rate at which material enters a vertex = rate at which it leaves the vertex
  – (same as Kirchhoff’s Current Law)
Flow Networks: Intro

• *Maximum flow problem* (in simple form)

  – What is the greatest rate at which material can be sent (shipped) from the source to the sink without violating any capacity constraints?

• Can be solved by efficient algorithms
Definitions etc.

• **Flow network** \( G=(V,E) \) is a directed graph
  – each edge \( (u,v) \) in \( E \) has a nonnegative **capacity** \( c(u,v) \geq 0 \)
  – if \( (u,v) \) is not in \( E \) then we assume \( c(u,v)=0 \)
  – two special vertices, source, \( s \), and sink, \( t \)
  – assume every vertex is on some \( s-t \) path

• Example: CLRS, Fig 26.1(a)
Example

Fig 26.1 (a) in CLRS
Definitions etc.

- Let $G=(V,E)$ be a flow network with capacity function $c$, source $s$ and sink $t$.
- A flow in $G$ is a real-valued function $f: V \times V \rightarrow \mathbb{R}$ that satisfies 3 properties:
  1. For all $u, v \in V$, $f(u,v) \leq c(u,v)$ - capacity constraint.
  2. For all $u, v \in V$, $f(u,v) = -f(v,u)$ - skew symmetry.
  3. For all $u \in V-\{s,t\}$,
     $$\sum_{v \in V} f(u, v) = 0$$ - flow conservation.
Definitions etc.

• The *net flow* from vertex \( u \) to vertex \( v \) is the quantity \( f(u,v) \) which can be +ve or –ve.

• The *value* of a flow \( f \) is

\[
|f| = \sum_{v \in V} f(s,v)
\]

(total net flow out of \( s \))

• In the *maximum-flow problem*, given \( G \) with \( s \) and \( t \), we wish to find a flow of maximum value from \( s \) to \( t \).
Network Flow: An Example

Fig 26.1 (b) in CLRS
A flow with value $|f| = 19$
Multiple Sources, Sinks?

• What if there are > 1 sources and sinks?
  – E.g., company with $m$ factories $n$ warehouses
• Can reduce to an ordinary maximum-flow problem
  – Can add a supersource and a supersink
  – E.g., Fig. 26.2, p. 648 in CLRS
  – Can prove the two problems are equivalent
Multiple Sources, Sinks?

Fig 26.2 in CLRS
Ford-Fulkerson Method

- Solves the maximum-flow problem
  - Involves several implementations/algorithms
- Depends on 3 (broad) ideas
  - Residual networks
  - Augmenting paths
  - Cuts
- And the max-flow min-cut theorem
Ford-Fulkerson Method

• Iterative method
  – Initialize \( f(u,v)=0 \) for all \( u, v \) in \( V \)
  – At each iteration, increase the flow by finding an augmenting path
    • (an \( s-t \) path along which we can push more flow)
  – Then augment the flow along this path
  – Repeat until no augmenting path is found
Ford-Fulkerson Method

FORD-FULKERSON-METHOD(G,s,t)

initialize flow $f$ to 0

while there exists an augmenting path $p$
    augment flow $f$ along $p$

return $f$

• Upon termination, yields a maximum flow
Residual Networks

• Given a flow network and a flow, the residual network consists of edges that can admit more net flow

• More formally
  – Suppose flow network \( G=(V,E) \), \( s, t \) are given
  – Let \( f \) be a flow in \( G \) and \( u, v \) be vertices in \( V \)
  – Additional net flow we can push from \( u \) to \( v \) before exceeding capacity \( c(u,v) \) is the residual capacity of \( (u,v) \) given by
    \[
    c_f(u,v) = c(u,v) - f(u,v)
    \]
Residual Networks

• Example
  – If $c(u,v)=16$ and $f(u,v)=11$ then we can ship
    $c_f(u,v)=5$ more units of flow

• When the net flow is $-ve$, $c_f(u,v)> c(u,v)$
  – Example: If $c(u,v)=16$ and $f(u,v)= -4$ then
    $c_f(u,v)=20$
  – This means: push 4 units $u \rightarrow v$ to cancel the 4 units net flow from $v \rightarrow u$, then push 16 more
Residual Networks

• Given a flow network $G=(V,E)$ and a flow $f$, the \textit{residual network} of $G$ induced by $f$ is $G_f=(V,E_f)$ where

$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

– Each edge of the residual network (residual edge) can admit a positive net flow
– Example: Fig. 26.3(b) on p. 652 in CLRS
Example Residual Network

Network ➔

Residual Network ➔
Residual Networks

• Important property (Lemma 26.2 in CLRS)
  – Let $G=(V,E)$, $s$, $t$ be a flow network and $f$ be a flow in $G$
  – Let $G_f$ be the residual network induced by $f$
  – Let $f^*$ be a flow in $G_f$
  – Then the flow sum $f+f^*$ is a flow in $G$ with value $|f+f^*| = |f|+|f^*|$

• Shows how a flow in $G_f$ relates to one in $G$
Augmenting Paths

- Given a flow network $G=(V,E)$ and a flow $f$, an **augmenting path** $p$ is a simple path from $s$ to $t$ in the residual network $G_f$
  - Each edge $(u,v)$ on the augmenting path admits some additional positive net flow from $u$ to $v$ without violating the capacity constraint
- Example
  - Fig. 26.3(b) on p. 652 in CLRS
Example Augmenting Path

Network

Residual network with augmenting path shaded

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Augmenting Paths

- **Residual capacity** $c_f(p)$ of an augmenting path $p$ is the maximum amount of net flow that we can ship along the edges of $p$

  $$c_f(p) = \min \{ c_f(u,v) : (u, v) \text{ is on } p \}$$
Augmenting Paths

- Important properties (Lemma 26.3 and Corollary 26.4 in CLRS)
  - Let $G=(V,E)$ be a flow network, $f$ a flow in $G$
  - If $f_p$ is a flow defined on an augmenting path $p$ of a residual network $G_f$ of $G$
  - If we add $f_p$ to $f$, we get another flow in $G$ whose value is closer to the maximum

- Example:
  - Fig. 26.3(c) on p. 652 in CLRS
Is there an augmenting path? → No
What does it mean? → Max flow attained
“Cuts” of Flow Networks

• A cut $(S,T)$ of flow network $G=(V,E)$ is a partition of $V$ into $S$ and $T=V-S$ such that the source $s$ is in $S$ and the sink $t$ is in $T$.

• If $f$ is a flow, then the net flow across the cut $(S,T)$ is $f(S,T)$ and the capacity of the cut $(S,T)$ is $c(S,T)$. 
“Cuts” of Flow Networks

- If \((S, T)\) is a cut in a flow network then
  - the **net flow** \(f(S, T)\) across the cut \((S, T)\) is
    \[
    f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)
    \]
  - the **capacity** \(c(S, T)\) across the cut \((S, T)\) is
    \[
    c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)
    \]

What is a **minimum cut**?
Example

Fig 26.4 in CLRS

Net flow across the cut is $f(S, T) = 19$

Capacity across the cut is $c(S, T) = 26$
Example – Minimum Cut

Max possible flow through the cut = $12 + 7 + 4 = 23$
Network has a capacity of at most 23
This is a minimum cut
“Cuts” of Flow Networks

• Property (Lemma 26.5)
  – Let $f$ be a flow in a flow network $G$ and let $(S,T)$ be a cut of $G$. Then the net flow across $(S,T)$ is $f(S,T) = |f|$.

• Corollaries
  – The value of a flow is the net flow into the sink.
  – The value of any flow $f$ in a flow network $G$ is bounded from above by the capacity of any cut of $G$. 
The net flow across any cut is the same and equal to the flow of the network $|f| = 23$
Max-flow Min-cut Theorem

• If $f$ is a flow in a flow network $G=(V,E)$ with source $s$ and sink $t$, then the following are equivalent
  – $f$ is a maximum flow in $G$
  – The residual network $G_f$ contains no augmenting paths
  – $|f| = c(S,T)$ for some cut $(S,T)$ of $G$
Basic Ford-Fulkerson Alg.

\begin{algorithm}
\textbf{FORD-FULKERSON}(G, s, t)
\begin{algorithmic}
1 \textbf{for each edge} \((u, v) \in E[G]\)
2 \hspace{1em} \textbf{do} \hspace{1em} f[u, v] \leftarrow 0
3 \hspace{1em} f[v, u] \leftarrow 0
4 \textbf{while} \hspace{1em} \text{there exists a path} \hspace{1em} p \hspace{1em} \text{from} \hspace{1em} s \hspace{1em} \text{to} \hspace{1em} t \hspace{1em} \text{in the residual network} \hspace{1em} G_f
5 \hspace{1em} \textbf{do} \hspace{1em} c_f(p) \leftarrow \min \{c_f(u, v) : (u, v) \text{ is in } p\}
6 \hspace{1em} \textbf{for each edge} \hspace{1em} (u, v) \hspace{1em} \text{in} \hspace{1em} p
7 \hspace{3em} \textbf{do} \hspace{3em} f[u, v] \leftarrow f[u, v] + c_f(p)
8 \hspace{3em} f[v, u] \leftarrow -f[u, v]
\end{algorithmic}
\end{algorithm}
Example

Original Network

Flow Network

Resulting Flow = 4
Example

Flow Network

Resulting Flow = 4

Residual Network

augmenting path

Resulting Flow = 11

Flow Network

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Example

Flow Network

Residual Network

Flow Network

Resulting Flow = 11

Resulting Flow = 19
Example

Flow Network

Residual Network

Flow Network

Resulting Flow = 19

augmenting path

Resulting Flow = 23
Example

Resulting Flow = 23

No augmenting path:
Maxflow=23

Residual Network
Analysis

\textsc{ford-fulkerson}(G, s, t)
1 for each edge \((u, v) \in E[G]\)
2 \hspace{1em} do \(f[u, v] \leftarrow 0\)
3 \hspace{1em} \(f[u, v] \leftarrow 0\)
4 \hspace{1em} while there exists a path \(p\) from \(s\) to \(t\) in the residual network \(G_f\)
5 \hspace{1em} do \(c_f(p) \leftarrow \min \{c_f(u, v) : (u, v) \text{ is in } p\}\)
6 \hspace{1em} for each edge \((u, v)\) in \(p\)
7 \hspace{1em} do \(f[u, v] \leftarrow f[u, v] + c_f(p)\)
8 \hspace{1em} \(f[v, u] \leftarrow -f[u, v]\)

\(O(E)\)
Analysis

• If capacities are all integer, then each augmenting path raises $|f|$ by $\geq 1$
• If max flow is $f^*$, then need $\leq |f^*|$ iterations
  – Running time is $O(E |f^*|)$
  – This is not polynomial in input size
  – Depends on $|f^*|$, which is not a function of $|V|$ or $|E|$
Additional Material

• On network flows, Ford-Fulkerson method and applications
  – Prof. Kincaid’s slides:
    – Slide set 1, Slide set 2

• Read, explore further for
  – Improvements over Ford-Fulkerson approach
    • Edmonds-Karp algorithm
    • Push-relabel algorithm
Conclusion

• We discussed
  – Flow networks
  – Residual networks, augmenting paths, cuts
  – *Max-flow min-cut theorem*
  – Ford-Fulkerson method

• End of discussion on Graph algorithms

• Next time
  – Computational geometry
References

• The lecture slides are based on the slides prepared by Prof. Sanath Jayasena for this class in previous years.

• CLRS book, 2e, Part VI: Graph Algorithms
  – Chapter 26: Maximum Flow

• Other references
  – Prof. Kincaid: http://www.math.wm.edu/~rrkinc/
  – Prof. James Elder
    http://elderlab.yorku.ca/~elder/teaching/cse3101/