GENERALIZED DERIVATIONS OF LIE ALGEBRAS

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Abstract. Suppose L is a finite-dimensional Lie algebra with multiplication $\mu\colon L \wedge L \to L$. Let $\Delta(L)$ denote the set of triples (f, f', f''), with $f, f', f'' \in \mathrm{Hom}(L, L)$, such that $\mu \circ (f \wedge I_L + I_L \wedge f') = f'' \circ \mu$. We consider the Lie algebra $\mathrm{GenDer}(L) = \{f \in \mathrm{Hom}(L, L) \mid \exists f', f''\colon (f, f', f'') \in \Delta(L)\}$. Well-researched subalgebras of $\mathrm{GenDer}(L)$ include the derivation algebra, $\mathrm{Der}(L) = \{f \in \mathrm{Hom}(L, L) \mid (f, f, f) \in \Delta(L)\}$, and the centroid, $\mathrm{C}(L) = \{f \in \mathrm{Hom}(L, L) \mid (f, 0, f) \in \Delta(L)\}$. We now study the subalgebra $\mathrm{QDer}(L) = \{f \in \mathrm{Hom}(L, L) \mid \exists f'\colon (f, f, f') \in \Delta(L)\}$, and the subspace $\mathrm{QC}(L) = \{f \in \mathrm{Hom}(L, L) \mid (f, -f, 0) \in \Delta(L)\}$. In characteristic $\neq 2$, $\mathrm{GenDer}(L) = \mathrm{QDer}(L) + \mathrm{QC}(L)$ and we are concerned with the inclusions $\mathrm{Der}(L) \subseteq \mathrm{QDer}(L)$ and $\mathrm{C}(L) \subseteq \mathrm{QC}(L) \cap \mathrm{QDer}(L)$. If $\mathrm{Z}(L) = 0$ then $\mathrm{C}(L) = \mathrm{QC}(L) \cap \mathrm{QDer}(L)$ and, under reasonable conditions on Lie algebras with toral Cartan subalgebras, we show $\mathrm{QDer}(L) = \mathrm{Der}(L) + \mathrm{C}(L)$; if L is a parabolic subalgebra of a simple Lie algebra of rank > 1 in characteristic 0, then we even have $\mathrm{GenDer}(L) = \mathrm{ad}(L) + (I_L)$. In general $\mathrm{QC}(L)$ is not closed under composition or Lie bracket; however, if $\mathrm{Z}(L) = 0$ then $\mathrm{QC}(L)$ is a commutative, associative algebra, and we describe conditions that force $\mathrm{QC}(L) = \mathrm{C}(L)$ or, equivalently, $\mathrm{GenDer}(L) = \mathrm{QDer}(L)$.

We show that, in characteristic 0, $\mathrm{GenDer}(L)$ preserves the radical of L, thus generalizing the classical result for $\mathrm{Der}(L)$.

We also discuss some applications of the main results to the study of functions $f \in \mathrm{Hom}(L, L)$ such that $f \circ \mu$ or $\mu \circ (f \wedge I_L)$ defines a Lie multiplication.

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1. Introduction

Let A be nonassociative algebra over a field of characteristic $\neq 2$ with multiplication $(a,b) \mapsto \mu(a,b)$. A *derivation* of A is a element $f \in \mathrm{Hom}(A,A)$ such that

$$\mu(f(x),y) + \mu(x,f(y)) = f(\mu(x,y))$$

for all $x,\ y \in A$. The set $\mathrm{Der}(A)$ of derivations of A is a well-studied Lie subalgebra of $\mathfrak{gl}(A)$. In this paper, we investigate a natural generalization of derivations.

We call $f \in \mathrm{Hom}(A,A)$ a *generalized derivation* of A if there exist elements $f',\ f'' \in \mathrm{Hom}(A,A)$ such that,

$$\mu(f(x),y) + \mu(x,f'(y)) = f''(\mu(x,y)) \tag{1.1}$$

for all $x,\ y \in A$, and we call $f \in \mathrm{Hom}(A,A)$ a *quasiderivation* of A if there exists $f' \in \mathrm{Hom}(A,A)$ such that

$$\mu(f(x),y) + \mu(x,f(y)) = f'(\mu(x,y))$$

for all $x,\ y \in A$. The set of generalized derivations of A forms a Lie subalgebra of $\mathfrak{gl}(A)$ which we denote by $\mathrm{GenDer}(A)$. The Lie subalgebra consisting of quasiderivations is then denoted by $\mathrm{QDer}(A)$.

Another important subalgebra of $\mathrm{GenDer}(A)$ is the *centroid*, $\mathrm{C}(A)$, of A consisting of $f \in \mathrm{Hom}(A,A)$ such that

$$\mu(f(x),y) = \mu(x,f(y)) = f(\mu(x,y))$$

for all $x,\ y \in A$. The centroid is even closed under composition. It is easy to show that, for Lie algebras with trivial center, $\mathrm{C}(L)$ is commutative (see, e.g. [14]) and so is the largest commutative subring of $\mathrm{Hom}(L,L)$, containing the base field, over which L is an algebra.

For centerless Lie algebras L, we have the tower

$$L \simeq \mathrm{ad}(L) \subseteq \mathrm{Der}(L) \subseteq \mathrm{QDer}(L) \subseteq \mathrm{GenDer}(L) \subseteq \mathfrak{gl}(L), \tag{1.2}$$

where $\mathrm{ad}(L)$ is the algebra of inner derivations. We are particularly interested in conditions that guarantee the collapsing of some of these inequalities. There has been much study of the inclusion $\mathrm{ad}(L) \subseteq \mathrm{Der}(L)$ from this viewpoint (see, e.g., [5] and [15] and the bibliographies therein). The next inclusion of (1.2) is always strict, for we have

$$\mathrm{QDer}(L) \supseteq \mathrm{Der}(L) \oplus \mathrm{C}(L), \quad \text{(vector space direct sum)} \tag{1.3}$$

and $0 \neq I_L \in \mathrm{C}(L)$. The interesting pursuit is for conditions that force $\mathrm{Der}(L) + \mathrm{C}(L) = \mathrm{QDer}(L)$.

Also of interest is the subset of $\mathrm{GenDer}(A)$ consisting of $f \in \mathrm{Hom}(A,A)$ such that

$$\mu(f(x),y) = \mu(x,f(y))$$

for all $x,\ y \in A$ and we call this set the *quasicentroid* of A, denoting it by $\mathrm{QC}(A)$. It is easy to verify that

$$\mathrm{C}(A) \subseteq \mathrm{QDer}(A) \cap \mathrm{QC}(A). \tag{1.4}$$

and, for skew-commutative or commutative A, that

$$\mathrm{GenDer}(A) = \mathrm{QDer}(A) + \mathrm{QC}(A). \tag{1.5}$$

(see Proposition 3.3). In his study of Levi factors in derivation algebras of nilpotent Lie algebras, Benoist [1] required the fact that $\mathrm{GenDer}(S) = \mathrm{QDer}(S) + \mathrm{C}(S)$ when S is a semisimple Lie algebra in characteristic 0. He established this by showing $\mathrm{QC}(S) = \mathrm{C}(S)$ for such algebras. This equality does not hold for general Lie algebras. Indeed, the quasicentroid need not be closed under composition or Lie multiplication. This motivates a study of the structure of $\mathrm{QC}(L)$ with particular reference to conditions that guarantee equality in the inclusion.

$$\mathrm{C}(L) \subseteq \mathrm{QC}(L). \tag{1.6}$$

We assume throughout this paper that algebras are finite-dimensional.

In Section 2, we recall some definitions and introduce notation.

Section 3 contains elementary observations about generalized derivations, some of which are technical results to be used later. The section also includes a characterization of Lie algebras L for which $\mathrm{GenDer}(L) = \mathfrak{gl}(L)$. For such algebras we even have $\mathrm{QDer}(L) = \mathfrak{gl}(L)$; this includes 3-dimensional simple algebras.

In Section 4, we examine the structure of $\mathrm{QDer}(L)$ for centerless Lie algebras with a toral Cartan subalgebra T. We first study $\mathrm{QDer}(L)_0$, which consists of those quasiderivations f for which $T \cdot f = 0$. For example, if L is solvable, then $\mathrm{QDer}(L)_0 \subseteq \mathrm{Der}(L) + \mathrm{C}(L)$. In general, $\mathrm{QDer}(L) \neq \mathrm{Der}(L) + \mathrm{C}(L)$ as is seen when L is 3-dimensional simple. That example, however, characterizes the obstruction to equality for parabolic subalgebras of semisimple Lie algebras in characteristic 0; namely, $\mathrm{QDer}(L) = \mathrm{Der}(L) + \mathrm{C}(L)$ if and only if the summands in the split form have rank > 1. More generally, we go on to show that equality holds if the algebra is generated by *special* weight spaces (see Section 2 for definition).

Section 5 has several results dealing with $\mathrm{QC}(L)$ for centerless Lie algebras L. For such algebras, we show that $\mathrm{QC}(L)$ is closed under composition and is, in fact, commutative. Our analysis makes strong use of the Fitting decomposition of $\mathrm{QC}(L)$ with respect to a Cartan subalgebra H of L: the Fitting 0-component is precisely $\mathrm{C}(L)$ while the Fitting 1-component maps L into $\mathrm{Z}(H)$ and maps H to 0. We show that $\mathrm{QC}(L) = \mathrm{C}(L) \oplus A$ in which A is an (associative-algebra) ideal and $A^2 = 0$. The information obtained on the structure of $\mathrm{QC}(L)$ also reveals conditions that force equality in (1.6). For example, equality holds: if $\mathrm{ad}_L(H)$ does not contain elements that are nilpotent of index 2 (in particular, when H is a torus); or if $L = [L, L]$. We show also that equality holds in (1.4) for centerless Lie algebras. Thus, by

(1.5) the algebras for which $\mathrm{QC}(L) = \mathrm{C}(L)$ are precisely those for which $\mathrm{QDer}(L) = \mathrm{GenDer}(L)$. Semisimple elements of $\mathrm{QC}(L)$ necessarily lie in $\mathrm{C}(L)$, so it is especially interesting to study the nilpotent elements of $\mathrm{QC}(L)$ and we show that these always map L into the nilradical; in fact, with some restrictions on the field, we show that, if $f \in \mathrm{QC}(L)$ is nilpotent of index m then $f(L)$ generates a nilpotent ideal of index m.

Section 6 is devoted to showing that, in characteristic 0, generalized derivations of L preserve the solvable radical of L thus generalizing the classical result for derivations.

In Section 7, we deal with applications of our analyses to two natural actions of $\mathfrak{gl}(V)$ on Lie algebra structures on V. Our study of $\mathrm{QC}(L)$ augments results of Ikeda on *projective doubles* of Lie algebras (L, μ), i.e., Lie algebras (L, ρ) with $\mathrm{ad}(L, \rho) \subseteq \mathrm{Der}(L, \mu)$. In particular, we look at *inner* projective doubles, wherein $\rho(x, y) = \mu(f(x), y)$ for some $f \in \mathrm{Hom}(L, L)$: for centerless L, $f \in \mathrm{Hom}(L, L)$ induces an inner projective double if and only if $f \in \mathrm{QC}(L)$ with $f^2 \in \mathrm{C}(L)$. We study also an analogous question of when $f \circ \mu$ is a Lie multiplication for nonsingular $f \in \mathrm{Hom}(L, L)$. This is the always the case if $f \in \mathrm{C}(L)$, and we call (L, μ) *robust* if $f \circ \mu$ is a Lie multiplication *only* when $f \in \mathrm{C}(L)$. We give a cohomological condition for robustness that is analogous to that for classical rigidity, though we point out the independence of the concepts. Our study of $\mathrm{QDer}(L)$ is relevant here and we show, for example, that parabolic subalgebras of split simple algebras of rank > 1 are robust.

The paper is interspersed with examples that emphasize the sharpness of the results.

We note that the term *generalized derivation* has been used with various technical meanings different from ours (see, e.g., [6, 7, 21]).

2. Definitions and Notation

We assume throughout that the characteristic of the ground field κ is not 2.

We shall be concerned chiefly with finite-dimensional Lie algebras, though some elementary results are valid for more general algebras. Let (A, μ) be a nonassociative algebra over κ. Thus, A is a vector space over κ, and $\mu\colon A \times A \to A$ is a bilinear map. It is also convenient at times to view the multiplication as a linear map $\mu\colon A \otimes A \to A$. We often write A in place of (A, μ). For $n \geq 1$, we define A^n to be the span of all products $x_1 \cdots x_n$, with $x_i \in A$, no matter how associated. The (two-sided) ideal of A generated by a set S will be denoted by $\langle S \rangle$. The *annihilator* of A is

$$\mathrm{Ann}(A, \mu) = \{x \in A \mid \mu(x, y) = \mu(y, x) = 0,\ \forall\, y\ \in A\}.$$

It is sometimes convenient to deal with the set of triples (f, f', f'') satisfying (1.1) and we denote this collection by $\Delta(A)$. Thus, $\Delta(A)$ is a Lie subalgebra of $\mathfrak{gl}(A) \times \mathfrak{gl}(A) \times \mathfrak{gl}(A)$.

If (A, μ) is a nonassociative algebra over κ and R is a commutative algebra over κ, we denote by $A[R]$ the algebra over κ whose underlying vector space is $A \otimes R$ with multiplication satisfying

$$(a_1 \otimes r_1) \cdot (a_2 \otimes r_2) = \mu(a_1, a_2) \otimes r_1 r_2,$$

for $a_1, a_2 \in A$, $r_1, r_2 \in R$.

Let L be a Lie algebra over κ. For $X, Y \subseteq L$, L a Lie algebra, let $\mathrm{Z}_X(Y)$ denote the *centralizer* of Y in X, i.e., $\{x \in X \mid [x, Y] = 0\}$; as usual, $\mathrm{Z}(L) = \mathrm{Z}_L(L)$ is the *center* of L; if $\mathrm{Z}(L) = 0$, we may say that L is *centerless*. If M and N are L-modules, the *transporter* of M to N is the ideal $\mathrm{Tr}(M, N) = \{x \in L \mid x \cdot M \subseteq N\}$; of special interest is the case when M and N are ideals in L and therefore modules with respect to the adjoint action. We denote by $\mathrm{Rad}(L)$ the (solvable) radical of L and by $\mathrm{NilRad}(L)$ the maximum nilpotent ideal of L. For $S \subseteq L$, $\langle S \rangle$ denotes the ideal generated by S. We say L is *directly indecomposable* if $L = H \oplus K$, a direct sum of ideals, implies H or K is 0.

An element, x, of a Lie algebra, L, is *semisimple* if $\mathrm{ad}(x)$ is a semisimple (diagonalizable over an extension field) element of $\mathrm{Hom}(L, L)$. A *torus*, T, is an abelian subalgebra of a Lie algebra, L, which consists of semisimple elements. We say T is *split* if, for $x \in T$, the characteristic roots of $\mathrm{ad}(x)$ belong to the base field. Unless otherwise indicated, the tori we consider will be split. We consider several naturally induced L-modules on which tori in L act diagonally; if M is such an L-module then, for $\alpha \in T^*$, α is a *weight* of T on M if the *weight space*, $M_\alpha := \{m \in M \mid t \cdot m = \alpha(t)m, \ \forall t \in T\} \neq 0$. We denote by $\mathcal{W}(M)$ the weights of T in M (T will always be clear in context)

For $\alpha \in \mathcal{W}(T)$, we set $L_{(\alpha)} := \oplus_{c \in \kappa} L_{c\alpha}$ and $L_{(\alpha)'} := \oplus_{\beta \notin \kappa\alpha} L_\beta$. We say a weight α for T in L is *special* if $L_{(\alpha)}$ does not contain a nontrivial ideal of L.

A *Heisenberg Lie algebra* of dimension $2n + 1$ has basis $\{x_i\}$ with $1 \leq i \leq 2n + 1$ and nonzero products, $[x_i, x_{n+i}] = -[x_{n+i}, x_i] = x_{2n+1}$ for $i = 1, \ldots, n$.

If L is a Lie algebra and M an L-module, $\mathrm{Z}^i(L, M)$, $\mathrm{B}^i(L, M)$, and $\mathrm{H}^i(L, M)$ denote the ith-cocycles, coboundaries and cohomology of L with coefficients in M, while δ denotes the coboundary operator. For more details, see [9, p.93 et seq]. We define $\mathring{L}$ to be the trivial L-module on the underlying vector space of L, while the adjoint module is still denoted by L. To distinguish the coboundary map δ on the complex $\mathrm{C}(L, L)$, we let $\mathring{\delta}$ denote the coboundary map on $\mathrm{C}(L, \mathring{L})$. Observe that $\mathrm{B}^2(L, \mathring{L}) = \mathrm{Hom}(L, L) \circ \mu$. Also, if $L = (L, \mu)$, $f \in \mathrm{Hom}(L, L)$, then $\delta(f \circ \mu) = \mathring{\delta}(f \circ \mu)$.

Let V be a vector space. The identity transformation of V is denoted I_V. For $S \subseteq V$, (S) denotes the linear span of S. We denote by $V \circledS V$ the symmetric product, and by $V \wedge V$ the exterior product; we may view these as subspaces of $V \otimes V$, where $V \circledS V = (v \otimes w + w \otimes v \mid v, w \in V)$ and $V \wedge V = (v \otimes w - w \otimes v \mid v, w \in V)$.

3. General Remarks

For any nonassociative algebra A, $\Delta(A)$ is an algebraic Lie algebra and so the semisimple and nilpotent components of generalized derivations are generalized derivations.

It is easy to see that generalized derivations preserve the annihilator of a nonassociative algebra, and hence the center of a Lie algebra. However, although $\mathrm{Der}(L)$ and $\mathrm{C}(L)$ clearly preserve the derived algebra of a Lie algebra L, neither $\mathrm{QDer}(L)$ nor $\mathrm{QC}(L)$ need do so: for quasiderivations, this is evident in the 2-dimensional, nonabelian Lie algebra spanned by x, y with $[x, y] = y$, the linear map satisfying $x \mapsto 0$, $y \mapsto x$ is a quasiderivation); see Example 5.7 for a Lie algebra L and $f_3 \in \mathrm{QC}(L)$ such that $f_3([L, L]) \not\subseteq [L, L]$.

Lemma 3.1. *Let A be a nonassociative algebra. Then*

(1) $\mathrm{QDer}(A)$ *is a Lie subalgebra of* $\mathrm{GenDer}(A)$.
(2) $[\mathrm{Der}(A), \mathrm{C}(A)] \subseteq \mathrm{C}(A)$.
(3) $[\mathrm{QDer}(A), \mathrm{QC}(A)] \subseteq \mathrm{QC}(A)$.
(4) $\mathrm{C}(A) \subseteq \mathrm{QDer}(A)$.
(5) $[\mathrm{QC}(A), \mathrm{QC}(A)] \subseteq \mathrm{QDer}(A)$.

Proof: (1), (2), (3) are immediate. For (4), observe that $f \in \mathrm{C}(A)$ implies $(f, f, 2f) \in \Delta(A)$. For (5), $f, g \in \mathrm{QC}(L)$ implies $([f, g], [f, g], 0) \in \Delta(A)$. □

Remark 3.2. If A is commutative or skew commutative then $(f, f', f'') \in \Delta(A)$ implies $(f', f, f'') \in \Delta(A)$.

Proposition 3.3. *Let A be a commutative or skew-commutative algebra, then*

(1) $\mathrm{GenDer}(A) = \mathrm{QDer}(A) + \mathrm{QC}(A)$.
(2) $\mathrm{QC}(A) + [\mathrm{QC}(A), \mathrm{QC}(A)]$ *is an ideal in the Lie algebra* $\mathrm{GenDer}(A)$.

Proof: For (1), we note that $(g, g', g'') \in \Delta(A)$ implies

$$(g, g', g'') = (f, f, g'') + (e, -e, 0),$$

where $f = (g + g')/2$, and $e = (g - g')/2$. With this, (2) follows from (3) and (5) of Lemma 3.1. □

Suppose $A = H \oplus K$, a direct sum of ideals, and $\mathrm{Ann}(A) = 0$. Then, if $f \in \mathrm{GenDer}(A)$, f must preserve H and K. Thus, we generalize the well known result for derivations to obtain the following useful result.

Lemma 3.4. *If $A = H \oplus K$ is the direct sum of ideals, and* $\mathrm{Ann}(A) = 0$, *then*

(1) $\mathrm{GenDer}(A) = \mathrm{GenDer}(H) \oplus \mathrm{GenDer}(K)$.
(2) $\mathrm{QDer}(A) = \mathrm{QDer}(H) \oplus \mathrm{QDer}(K)$.
(3) $\mathrm{C}(A) = \mathrm{C}(H) \oplus \mathrm{C}(K)$.

(4) $\mathrm{QC}(A) = \mathrm{QC}(H) \oplus \mathrm{QC}(K)$. □

The quasiderivations of an algebra can be embedded as derivations in a larger algebra. (That is their key role in [1].) For this, let A be a nonassociative algebra over κ and t an indeterminant. We define an algebra, $\breve{A}$, over κ , canonically associated to A, as $\breve{A} := A[t\kappa[t]/(t^3)]$. We write at (at^2) in place of $a \otimes t$ ($a \otimes t^2$). If U is a subspace of A such that $A = U \oplus A^2$ then

$$\breve{A} = At + At^2 = At + A^2t^2 + Ut^2,$$

Now we define a map $\iota_U \colon \mathrm{QDer}(A) \to \mathrm{Der}(\breve{A})$ so that, for $(f, f, f') \in \Delta(A)$,

$$\iota_U(f)(at + bt^2 + ut^2) = f(a)t + f'(b)t^2$$

for all $a \in A$, $b \in A^2$ and $u \in U$. Note that ι_U is injective and that $\iota_U(f)$ does not depend on the choice of f'.

Now, for any nonassociative algebra B, let

$$\mathrm{ZDer}(B) := \{f \in \mathrm{Hom}(B, B) \mid f(B) \subseteq \mathrm{Ann}(B),\ f(B^2) = 0\}$$

Then $\mathrm{ZDer}(B)$ is an ideal in $\mathrm{Der}(B)$.

The following proposition is implicit in [13, p.191].

Proposition 3.5. *If A is a nonassociative algebra with $\mathrm{Ann}(A) = 0$, and ι_U as above, then $\mathrm{Der}(\breve{A})$ is a semidirect sum:*

$$\mathrm{Der}(\breve{A}) = \iota_U(\mathrm{QDer}(A)) \oplus \mathrm{ZDer}(\breve{A}).$$

Proof: Observe that $\mathrm{Ann}(\breve{A}) = At^2$. So any linear $f \colon At + Ut^2 \to At^2$ extends to an element of $\mathrm{ZDer}(\breve{A})$ by taking $f(A^2t^2) = 0$. Thus, given any $g \in \mathrm{Der}(\breve{A})$ we can subtract an element of $f \in \mathrm{ZDer}(\breve{A})$ and thereby force $(g - f)(At) \subseteq At$ and $(g - f)(Ut^2) = 0$ (note that $g(Ut^2) \subseteq g(\mathrm{Ann}(\breve{A})) \subseteq \mathrm{Ann}(\breve{A})$). Also, since $\breve{A}^2 = A^2t^2$, $(g - f)(A^2t^2) \subseteq A^2t^2$. Thus $(h, h, h') \in \Delta(A)$ where $(g - f)(at) = h(a)t$, $(g - f)(bt^2) = h'(b)t$ for $a \in A$, $b \in A^2$, and so, $(g - f) = \iota_U(h) \in \iota_U(\mathrm{QDer}(A))$. □

Observe that, in the situation of Proposition 3.5, $\iota_U(\mathrm{QDer}(A))$ may also be viewed as the natural image of $\mathrm{Der}(\breve{A})$ in $\mathrm{Hom}(\breve{A}/\breve{A}^2, \breve{A}/\breve{A}^2)$.

If (A, μ) is an algebra and f in $\mathrm{Hom}(A, A)$, we define $f^\# \in \mathrm{Hom}(A \otimes A, A \otimes A)$ so that $f^\#(v \otimes w) = f(v) \otimes w + v \otimes f(w)$ for all $v, w \in A$. Note that $\mu \circ f^\#$ factors through $\mu(A \otimes A)$ (i.e., $\mu \circ f^\# = f' \circ \mu$), for some $f' \in \mathrm{Hom}(A, A)$), if and only if $f^\#(\mathrm{Ker}(\mu)) \subseteq \mathrm{Ker}(\mu)$.

Lemma 3.6. *Let (A, μ) be an algebra and $f \in \mathrm{Hom}(A, A)$. Then $f \in \mathrm{QDer}(A, \mu)$ if and only if $f^\#(\mathrm{Ker}(\mu)) \subseteq \mathrm{Ker}(\mu)$.*

Proof: $f \in \mathrm{QDer}(A)$ if and only if there exists $f' \in \mathrm{Hom}(A, A)$ such that $f' \circ \mu = \mu \circ f^\#$. □

Proposition 3.7. *If* (A, μ) *is an algebra with* $\mu \neq 0$*, and* $\mathrm{QDer}(A, \mu) = \mathfrak{gl}(A)$ *then*

(1) $\dim(A) \leq 3$.
(2) *If* $\dim(A) = 1$*, then* $(A, \mu) \cong \kappa$*. If* $\dim(A) = 2$*, then* A *is a solvable Lie algebra, and if* $\dim(A) = 3$*, then* (A, μ) *is a simple, skew commutative, nonassociative algebra.*

Conversely, if L *is one of these algebras,* $\mathrm{QDer}((A, \mu)) = \mathfrak{gl}(A)$.

Proof: $\mathfrak{gl}(A)$ acts on $A \otimes A$ via $f \cdot (v \otimes w) = f^{\#}(v \otimes w)$ for all $v, w \in A$ with $f^{\#}$ as above. By Lemma 3.6, $\mathfrak{gl}(A) = \mathrm{QDer}(A)$ implies $\mathfrak{gl}(A) \cdot \mathrm{Ker}(\mu) \subseteq \mathrm{Ker}(\mu)$. Now the only proper subspaces of $A \otimes A$, invariant under this action of $\mathfrak{gl}(A)$ are $A \wedge A$, and $A \circledS A$. Thus we have $\mu \colon A \otimes A \to A$ with kernel $0, A \wedge A$ or $A \circledS A$. Using $\dim(A) \geq \dim(A \otimes A) - \dim(\mathrm{Ker}(\mu))$, an easy computation shows that if $\mathrm{Ker}(\mu) = 0$ or $A \wedge A$, then $\dim(A) \leq 1$, while if $\mathrm{Ker}(\mu) = A \circledS A$, $\dim(A) \leq 3$.

If $\dim(A) = 1$, then $A \wedge A = 0$, so that the only possibility for $\mathrm{Ker}(\mu)$ is 0, and the only nontrivial, 1-dimensional algebra over a field is isomorphic to that field. If $\dim(A) = 2$ or 3, then the only possibility for $\mathrm{Ker}(\mu)$ is $A \circledS A$ so μ must be skew commutative. If $\dim(A) = 2$, then $\dim(\mu(A \otimes A)) = 1$, so that A is the 2-dimensional nonabelian Lie algebra. If $\dim(A) = 3$, then $\dim(\mu(A \otimes A)) = 3$, so μ must be surjective and it follows that A cannot have a proper ideal.

Noting the $\mathfrak{gl}(A)$-invariance of the kernel of μ for these algebras, the converse is straightforward . □

The next theorem makes use of Theorem 5.11 (whose proof is independent of this result).

Theorem 3.8. *Let* L *be a Lie algebra such that* $\mathrm{GenDer}(L) = \mathfrak{gl}(L)$*. Then* $\mathrm{QDer}(L) = \mathfrak{gl}(L)$*. Thus, either* L *is abelian, 2-dimensional solvable or 3-dimensional simple. Conversely, if* L *is one of these algebras,* $\mathrm{GenDer}(L) = \mathfrak{gl}(L)$.

Proof: Assume $\mathrm{GenDer}(L) = \mathfrak{gl}(L)$ and that $[L, L] \neq 0$.

Claim: $\mathrm{QC}(L) = (I_L)$. To see this, let $K := \mathrm{QC}(L) + [\mathrm{QC}(L), \mathrm{QC}(L)]$, the ideal of $\mathrm{GenDer}(L)$ generated by $\mathrm{QC}(L)$ (Proposition 3.3). Then K is an ideal of $\mathrm{GenDer}(L) = \mathfrak{gl}(L)$ containing I_L.

Case 1: $\mathrm{Z}(L) \neq 0$. Since $\mathrm{Z}(L) \neq L$ and K stabilizes $\mathrm{Z}(L)$, we get that $K = (I_L)$ since (I_L) is the only ideal of $\mathfrak{gl}(L)$ that stabilizes a proper subspace of L

Case 2: $\mathrm{Z}(L) = 0$. By Theorem 5.11, $\mathrm{QC}(L)$ is commutative and therefore K is an abelian ideal in $\mathfrak{gl}(L)$. It follows that $K = (I_L)$.

By the above claim and Proposition 3.1, $\mathrm{QDer}(L) + (I_L) = \mathfrak{gl}(L)$. But, since $(I_L, I_L, 2I_L) \in \Delta(L)$, we know $(I_L) \in \mathrm{QDer}(L)$. Therefore, $\mathrm{QDer}(L) = \mathfrak{gl}(L)$. The converse follows at once from Proposition 3.7. □

The following easy lemma provides a useful connection between quasiderivations and Lie algebra cohomology.

Lemma 3.9. *Let L be a Lie algebra. For $f, f' \in \mathrm{Hom}(L, L)$,*

(1) *$f \in \mathrm{QDer}(L)$ if and only if $\delta f \in B^2(L, \mathring{L})$.*
More specifically, $(f, f, f') \in \Delta(L)$ if and only if $\delta f = \mathring{\delta}(f' - f)$.

(2) *If $\mathring{\delta} f \in \mathrm{Z}^2(L, L)$, then, $\forall\, x, y, z \in L$,*

$$[x, f([y, z])] + [y, f([z, x])] + [z, f([x, y])] = 0.$$

In particular, if $(f, f, f') \in \Delta(L)$, then

$$[x, (f' - f)([y, z])] + [y, (f' - f)([z, x])] + [z, (f' - f)([x, y])] = 0.$$

□

4. Quasiderivations of Lie Algebras

Let L be a Lie algebra containing a (diagonalized) torus T. Then

$$\mathrm{QDer}(L) = \mathrm{QDer}(L)_0 \,\oplus\, T \cdot \mathrm{QDer}(L)$$

(the second summand is comprised of the nonzero weight spaces). We also take note of the subspace

$$\mathrm{QDer}_T(L) := \{f \in \mathrm{QDer}(L) \mid f(T) = 0\}.$$

Lemma 4.1. *Let L be a Lie algebra and $T \subseteq L$ a torus. Then*

$$\mathrm{QDer}(L) = \mathrm{QDer}(L)_0 + \mathrm{QDer}_T(L) + \mathrm{ad}(L).$$

Proof: Let $f \in \mathrm{QDer}(L)$. For $t, t' \in T$, $(t \cdot f)(t') = [f(t), t']$ so that $t \cdot f - \mathrm{ad}(f(t)) \in \mathrm{QDer}_T(L)$. Hence, $T \cdot \mathrm{QDer}(L) \subseteq \mathrm{QDer}_T(L) + \mathrm{ad}(L)$. □

4.1. The structure of $\mathbf{QDer(L)_0}$.

We consider the structure of $\mathrm{QDer}(L)_0$ for centerless Lie algebras.

Lemma 4.2. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$, a torus with $T = L_0$ and $\mathrm{Z}(L) = 0$. Then*

$$\mathrm{QDer}(L)_0 = \mathrm{QDer}_T(L)_0 + (I_L).$$

Proof: Let $f \in \mathrm{QDer}(L)_0$ and suppose $(f, f, f') \in \Delta(L)$. Let B be a maximal independent set in $\mathcal{W}(L)$. Since $T \cap \mathrm{Z}(L) = 0$, B is a basis of T^* and so there is a basis $\{t_\beta \mid \beta \in B\}$ of T dual to B. For each $\beta \in B$, let $0 \neq x_\beta \in L_\beta$. For $\beta, \gamma \in B$, we have

$$[f(t_\beta), x_\gamma] + [t_\beta, f(x_\gamma)] = f'([t_\beta, x_\gamma]).$$

Since $f \in \mathrm{QDer}(L)_0$,we have that $f(T) \subseteq T$, and $f(L_\gamma) \subseteq L_\gamma$, so that

$$\gamma(f(t_\beta))x_\gamma + \gamma(t_\beta)f(x_\gamma) = \gamma(t_\beta)f'(x_\gamma).$$

Thus, if $\beta \neq \gamma$, then $\gamma(f(t_\beta)) = 0$. It follows that $f(t_\beta) = c_\beta t_\beta$ for some scalar c_β. In particular, f acts diagonally on T. Thus,

$$T = \bigoplus_{c \in \Gamma} T_c,$$

where Γ is the set of characteristic roots for the action of f on T, and T_c is the root space for c. It suffices now to prove that $|\Gamma| = 1$, for, if $T = T_c$, then

$$f - cI_L \in \mathrm{QDer}_T(L)_0.$$

Claim (1): For any $0 \neq \alpha \in \mathcal{W}(L)$, there is a unique $c \in \Gamma$ such that $[T_c, L_\alpha] \neq 0$. Proof: Suppose that $[T_{c_1}, L_\alpha] \neq 0$ and $[T_{c_2}, L_\alpha] \neq 0$, with $c_1, c_2 \in \Gamma$. Then there exists $t_i \in T_{c_i}$, $i = 1, 2$, such that $\alpha(t_i) = 1$. Let $0 \neq x_\alpha \in L_\alpha$. For $i = 1, 2$, we have

$$[f(t_i), x_\alpha] + [t_i, f(x_\alpha)] = f'([t_i, x_\alpha]),$$

or

$$c_i x_\alpha + f(x_\alpha) = f'(x_\alpha).$$

Hence, $c_1 = c_2$, proving Claim (1).

For each $c \in \Gamma$, let $w(c) := \{\alpha \in \mathcal{W}(L) \mid [T_c, L_\alpha] \neq 0\}$.

Claim (2): If $\alpha_i \in w(c_i)$, for $i = 1, 2$, with $c_1 \neq c_2$, then $[L_{\alpha_1}, L_{\alpha_2}] = 0$. Proof: take $t_i \in T_{c_i}$ such that $\alpha_i(t_i) = 1$. By Claim (1), we also know that $\alpha_1(t_2) = \alpha_2(t_1) = 0$. In particular, $\alpha_1 + \alpha_2 \neq 0$, so that if $[L_{\alpha_1}, L_{\alpha_2}] \neq 0$, we would have $[L_{\alpha_1}, L_{\alpha_2}] \nsubseteq T$. But then, $[t_i, [L_{\alpha_1}, L_{\alpha_2}]] = [L_{\alpha_1}, L_{\alpha_2}]$, for $i = 1, 2$. By Claim (1), this can only happen if $[L_{\alpha_1}, L_{\alpha_2}] = 0$, thus proving Claim (2).

Now, for each $c \in \Gamma$, let

$$H_c := T_c + \sum_{\alpha \in w(c)} L_\alpha.$$

Then we have $L = \bigoplus_{c \in \Gamma} H_c$, a direct sum as vector spaces. But, by the above, $H_c = \mathrm{Z}_L(\sum_{c' \neq c} H_{c'})$ (equality holds since $\mathrm{Z}(L) = 0$). Thus each H_c is an ideal and $\bigoplus_{c \in \Gamma} H_c$ is an algebra direct sum. By the indecomposability of L, $|\Gamma| = 1$. □

Lemma 4.3. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$, a torus with $T = L_0$, $\mathrm{Z}(L) = 0$, and $\dim(T) > 1$. Then, if $0 \neq t \in T$, there exist independent weights $\alpha, \beta \in \mathcal{W}(L)$ such that $\alpha(t) \neq 0$ and $\beta(t) \neq 0$.*

Proof: Fix $\alpha \in \mathcal{W}(L)$ for which $\alpha(t) \neq 0$ (α exists since $T \cap \mathrm{Z}(L) = 0$) and assume $\beta(t) = 0$ for all $\beta \in \mathcal{W}(L)$, $\beta \notin (\alpha)$. Suppose $\beta \in \mathcal{W}(L) \setminus (\alpha)$ and $0 \neq \gamma \in \mathcal{W}(L) \cap (\alpha)$. Since $\beta + \gamma \notin (\alpha)$ but $\beta(t) + \gamma(t) = \gamma(t) \neq 0$, it follows that $\beta + \gamma \notin \mathcal{W}(L)$, so that $[L_\beta, L_\gamma] = 0$. Let

$$\widehat{T} = \{t' \in T \mid \alpha(t') = 0\},$$

so $T = (t) \oplus \widehat{T}$. Let

$$A = (t) + \sum_{0 \neq \beta \in (\alpha)} L_\beta,$$

and

$$B = \widehat{T} + \sum_{\beta \notin (\alpha)} L_\beta.$$

Clearly, A is a subalgebra of L, $L = A \oplus B$ as a vector space, and B centralizes A. Since $\mathrm{Z}(L) = 0$, B must be precisely $\mathrm{Z}_L(A)$ (the centralizer of A in L) so that B is a subalgebra of L and hence, both A and B are ideals. Thus, $L = A \oplus B$, is an algebra direct sum, contradicting the indecomposability of L. □

Lemma 4.4. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, $\mathrm{Z}(L) = 0$, and $\dim(T) > 1$. Then*

$$\mathrm{QDer}_T(L)_0 \subseteq \mathrm{Der}(L).$$

Proof: Let $f \in \mathrm{QDer}_T(L)_0$. For $t \in T$, $x \in L$, $[t, f(x)] = f'([t, x])$. Hence, if $x \in L_\alpha$, $\alpha(t)f(x) = \alpha(t)f'(x)$. Thus, if $\alpha \neq 0$, $f(x) = f'(x)$, so f and f' agree on $[T, L]$.

It suffices to show that $f'(T \cap [L, L]) = 0$, whence $f'|_{[L,L]} = f|_{[L,L]}$, so that $f \in \mathrm{Der}(L)$.

The space $T \cap [L, L]$ is spanned by products of the form $[x, y]$ where, for some $\alpha \neq 0$, $x \in L_\alpha$ and $y \in L_{-\alpha}$. Consider such x, y. By weights, $f'([x, y]) = [f(x), y] + [x, f(y)] \in L_0 = T$. Let now $z \in L_\beta$ for $\beta \neq \pm\alpha$. Using $f'([x, z]) = f([x, z])$ and $f'([y, z]) = f([y, z])$ and Lemma 3.9(2), we get $[z, (f' - f)([x, y])] = 0$, so that $[f'([x, y]), z] = [f([x, y]), z] = 0$. By Lemma 4.3, we must have $f'([x, y]) = 0$. □

These lemmas give us the major result of this subsection.

Theorem 4.5. *If L is a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, $\mathrm{Z}(L) = 0$, and $\dim(T) > 1$, Then*

$$\mathrm{QDer}(L)_0 = \mathrm{Der}(L)_0 + (I_L).$$

Proof: By Lemmas 4.1, 4.2, and 4.4, $\mathrm{QDer}(L)_0 \subseteq \mathrm{Der}(L)_0 \oplus (I_L)$. The reverse inclusion is clear. □

Corollary 4.6. *Let L be a directly indecomposable Lie algebra with $L \neq [L, L]$, $T \subseteq L$ a torus with $T = L_0$, $\mathrm{Z}(L) = 0$. Then*

$$\mathrm{QDer}(L)_0 = \mathrm{Der}(L)_0 \oplus (I_L).$$

Proof: By Theorem 4.5, we may assume $T = (t)$. Since $L \neq [L, L]$, $L = (t) \oplus [t, L]$ (vector space direct sum), and $t \notin [L, L]$. Let $[t, L] = \sum_{\lambda \neq 0} L_\lambda$ with $[t, x] = \lambda x$ for $x \in L_\lambda$. Take $(f, f, f') \in \Delta(L)$ with $f \in \mathrm{QDer}(L)_0$. By Lemma 4.2, we may assume $f(t) = 0$. Since f' may be arbitrarily redefined on a fixed complement of $[L, L]$, we may assume $f'(t) = 0$. Let $x \in L_\lambda$ with $\lambda \neq 0$. Since $f \in \mathrm{QDer}(L)_0$, $f(x) \in L_\lambda$, so

$$\lambda f'(x) = f'([t, x]) = [t, f(x)] = \lambda f(x).$$

Hence, $f = f'$ and so $f \in \mathrm{Der}(L)_0$. □

Corollary 4.7. *Let L be a solvable Lie algebra, $T \subseteq L$ a (not necessarily diagonalized) torus with $T = L_0$, $\mathrm{Z}(L) = 0$. Then*

$$\mathrm{QDer}(L)_0 = \mathrm{Der}(L)_0 \oplus \mathrm{C}(L),$$

and this is a direct sum as algebras. In fact, $\mathrm{C}(L) = \mathrm{Z}(\mathrm{GenDer}(L))$.

Proof: We may assume that the base field is algebraically closed and, thus, that T is diagonalized. (Note that the 0-weight space is defined even if T is not diagonalized.) Let $L = \bigoplus_i L_i$ with each L_i directly indecomposable. Then T splits accordingly and, by Corollary 4.6, $\mathrm{QDer}(L_i)_0 = \mathrm{Der}(L_i)_0 \oplus (I_{L_i}) \subseteq \mathrm{Der}(L)_0 \oplus \mathrm{Z}_{\mathrm{C}(L_i)}(\mathrm{GenDer}(L_i))$. Thus, using Lemma 3.4,

$$\begin{aligned}\mathrm{QDer}(L)_0 &\subseteq \mathrm{Der}(L)_0 \oplus \mathrm{Z}_{\mathrm{C}(L)}(\mathrm{GenDer}(L))\\ &\subseteq \mathrm{Der}(L)_0 \oplus \mathrm{C}(L)\\ &\subseteq \mathrm{QDer}(L)_0.\end{aligned}$$

Hence, we have equality throughout and $\mathrm{C}(L) \subseteq \mathrm{Z}(\mathrm{GenDer}(L))$. But $\mathrm{Z}(\mathrm{GenDer}(L)) \subseteq \mathrm{Z}_{\mathfrak{gl}(L)}(\mathrm{ad}(L)) = \mathrm{C}(L)$. □

4.2. **Consequences of generation by special weight spaces.** Recall that a weight $\alpha \in \mathcal{W}(L)$ is called special if $L_{(\alpha)}$ does not contain a nontrivial ideal of L. Note that $[L_{(\alpha)}, L_{(\alpha)'}] \subseteq L_{(\alpha)'}$.

Lemma 4.8. *If $T \subseteq L$ is a torus and α is a special weight for T on L, then*

1. (1) $\dim(T) > 1$.
2. (2) *If M is a direct summand of L such that $L_\alpha \cap M \neq 0$, then α is a special weight for $T \cap M$ in M.*

Proof: To see (1), we need only note that, if α is a special weight, then there must exist a weight which is linearly independent of α. Since an ideal generated by an element of a direct summand stays in the summand, (2) is clear. □

Lemma 4.9. *Let A, B be subspaces of a Lie algebra L such that $L = A+B$, $[A,B] \subseteq B$. If $x \in A$ is such that $[x,B] = 0$, then $[L^n, x] = [A^n, x]$ for all $n \geq 1$.*

Proof: We note first that $[B,[A^n,x]] = 0$ for all $n \geq 1$. This follows by induction using

$$\begin{aligned}[B,[A^n,x]] &= [B,[A,[A^{n-1},x]]]\\ &= [[B,A],[A^{n-1},x]] + [A,[B,[A^{n-1},x]]]\\ &\subseteq [B,[A^{n-1},x]] + [A,[B,[A^{n-1},x]]]\end{aligned}$$

From this, the lemma also follows by induction using

$$[L^n, x] = [A, [L^{n-1}, x]] + [B, [L^{n-1}, x]]$$

□

Lemma 4.10. *Let $T \subseteq L$ be a torus and $\alpha \in \mathcal{W}(L)$. Then α is special if and only if, for all $0 \neq x \in L_{(\alpha)}$, $[x, L_{(\alpha)'}] \neq 0$.*

Proof: For $x \in L_{(\alpha)}$, $[x, L_{(\alpha)'}] \neq 0$ implies $[x, L_{(\alpha)'}] \subseteq \langle x \rangle \cap L_{(\alpha)'} \neq 0$, so the sufficiency is clear.

For the necessity, put $A = L_{(\alpha)}$, $B = L_{(\alpha)'}$ in Lemma 4.9. If $0 \neq x \in A$, with $[x, B] = 0$, then $\langle x \rangle = (x) + \sum_n [L^n_{(\alpha)}, x] \subseteq L_{(\alpha)}$, whence α is not special.

□

Lemma 4.11. *Let L be a Lie algebra, and $T \subseteq L$ a torus with $T = L_0$. Suppose that $\mathrm{Z}(L) = 0$ and that L is generated by special weight spaces. Then, for any $\alpha \neq 0$,*

$$\mathrm{QDer}_T(L)_\alpha = 0.$$

Equivalently, $T \cdot \mathrm{QDer}_T(L) = 0$.

Proof: By Lemma 4.8, if L is generated by special weight spaces, then this is true for any direct summand of L. Thus, we may assume that L is indecomposable. Now take $f \in \mathrm{QDer}_T(L)_\alpha$ with $\alpha \neq 0$. Let $t \in T$, $0 \neq \beta \in \mathcal{W}(L)$, and $0 \neq x_\beta \in L_\beta$. Since $f(T) = 0$, $[t, f(x_\beta)] = f'([t, x_\beta])$ so we have

$$(\alpha(t) + \beta(t)) f(x_\beta) = \beta(t) f'(x_\beta). \tag{4.1}$$

Thus, if $f(x_\beta) \neq 0$, α is a multiple of β. Also, for $\beta \neq 0, -\alpha$, $f(x_\beta) \neq 0$ if and only if $f'(x_\beta) \neq 0$. In particular, $f(L_{(\alpha)'}) = f'(L_{(\alpha)'}) = 0$.

Equation (4.1) also gives $f'(L_{-\alpha}) = 0$. We show furthermore that $f(L_{-\alpha}) = 0$. If $x \in L_{-\alpha}$, then $f(x) \in T$ by weights. However, for any $\beta \notin (\alpha)$, $[f(x), L_\beta] \subseteq [x, f(L_\beta)] + f'(L_{-\alpha+\beta}) = 0$, so $f(x) = 0$ by Lemma 4.3. Note also, that this implies $f(L) \cap T = 0$. In addition, we now have, for any $\beta \neq 0$, and $x \in L_\beta$, that $f(x) \neq 0$ if and only if $f'(x) \neq 0$.

Let $H := T + L_{(\alpha)}$. Then H is a subalgebra of L and $f(L) \subseteq L_{(\alpha)}$.

Let $K := \mathrm{Nullspace}(f)$. Since $f \in \mathrm{QDer}_T(L)_\alpha$, K is T-invariant, whence K is the direct sum of T weight spaces. We claim that K is a subalgebra of L. For this, take $x \in K_\beta$, $y \in K_\gamma$ with $\beta \neq 0, \gamma \neq 0$. If $\beta + \gamma = 0$, then $[x, y] \in T \subseteq K$. If $\beta + \gamma \neq 0$, we have $f'[x, y] = [f(x), y] + [x, f(y)] = 0$, so $f[x, y] = 0$, and again, $[x, y] \in K$.

Since $K \supseteq L_{(\alpha)'}$, we have $L = H + K$.

Next, since $[L_{(\alpha)}, L_{(\alpha)'}] \subseteq L_{(\alpha)'}$,

$$[f(L), L_{(\alpha)'}] = [f(L_{(\alpha)}), L_{(\alpha)'}] \subseteq [L_{(\alpha)}, f(L_{(\alpha)'})] + f'(L_{(\alpha)'}) = 0.$$

Finally, since L is generated by special weight spaces, we need only show that these weight spaces are in the subalgebra K. For this, all that remains is to show that $f(L_\gamma) = 0$ if $\gamma \in (\alpha)$ is special. If $x \in L_\gamma$ then $f(x) \in$

$L_{(\alpha)} = L_{(\gamma)}$ and $[f(x), L_{(\gamma)'}] = [f(x), L_{(\alpha)'}] = 0$. But then, by Lemma 4.10, $f(x) \neq 0$ would contradict the specialness of γ. □

As a result of Lemma 4.2, Theorem 4.5, and Lemma 4.11, we have:

Theorem 4.12. *Let L be a directly indecomposable Lie algebra, $T \subseteq L$ a torus with $T = L_0$, and $\mathrm{Z}(L) = 0$. Suppose L is generated by special weight spaces. Then*

$$\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus (I_L).$$

□

Corollary 4.13. *If L is a parabolic subalgebra of a split, simple Lie algebra of rank > 1 over a field of characteristic 0, then*

$$\mathrm{QDer}(L) = \mathrm{ad}(L) \oplus (I_L).$$

Proof: The parabolic subalgebras are complete [11, 20]. □

Corollary 4.14. *Let L be a Lie algebra of dimension > 3 in characteristic > 3. Suppose further that $\mathrm{C}(L) = (I_L)$ and that L has a nonsingular trace form on some representation. Then*

$$\mathrm{QDer}(L) = \mathrm{ad}(L) \oplus (I_L).$$

Proof: (We outline the underlying ideas and refer the reader to [19, chapters II, V] for necessary background details.) We may assume the base field is algebraically closed. The trace-form assumption implies L is classical [19, p. 28] (this uses the assumption characteristic > 3) and the fact that $\mathrm{C}(L) = (I_L)$ then implies L is simple. By a theorem of Kaplansky [10, Theorem 3], $\dim(L) > 3$ implies $\mathrm{rank}(L) > 1$ for such algebras and so the simplicity of L implies all roots are special. Finally, Block [2] has shown that $\mathrm{Der}(L) = \mathrm{ad}(L)$ for these algebras. □

Remark 4.15. Corollary 4.14 is an extension of the result of Hopkins [7] that these algebras have no nontrivial *antiderivations*, that is, $f \in \mathrm{Hom}(L, L)$ such that $(f, f, -f) \in \Delta(L)$. This follows from the above since $\mathrm{Der}(L) + (I_L)$ contains no antiderivations in characteristic > 3.

Corollary 4.16. *Let L be a Lie algebra as in Theorem 4.12 except that L need not be indecomposable. Then*

$$\mathrm{QDer}(L) = \mathrm{Der}(L) \oplus \mathrm{C}(L), \text{ direct sum as Lie algebras.}$$

Proof: By Lemma 3.4 and Theorem 4.12, we have $\mathrm{QDer}(L) \subseteq \mathrm{Der}(L) + Z_{\mathrm{C}(L)}(\mathrm{QDer}(L)) \subseteq \mathrm{Der}(L) + \mathrm{C}(L) \subseteq \mathrm{QDer}(L)$. Since it is clear that $\mathrm{Der}(L) \cap \mathrm{C}(L) = 0$ for centerless Lie algebras, the result follows. □

Remark 4.17. Corollary 4.16 shows that for such a Lie algebra, L, $\mathrm{C}(L) \subseteq \mathrm{Z}_{\mathrm{QDer}(L)}(\mathrm{Der}(L))$. Example 5.7 shows that this is not the case for general centerless Lie algebras.

Let $L = T + N$, with T a torus, N a nilpotent ideal, $\dim(T) = \dim(N/[N,N])$, and let U be a T-invariant complement of $[N,N]$ in N. Write $U = \oplus_\alpha U_\alpha$, where the U_α are 1-dimensional weight spaces for T. It is well known that one can associate a graph $\Gamma(T,N)$ to $T+N$ as follows: vertices of $\Gamma(T,N)$ are the weights of T on U while an edge joins the weights α and β if and only if $\alpha+\beta$ is a weight. Since U is isomorphic to $N/[N,N]$ as a T-module, $\Gamma(T,N)$ does not depend on the choice of U. (See, for example, [12] where we associate Lie algebras to arbitrary graphs.)

Corollary 4.18. *Suppose that $L = T + N$ is a semi-direct sum with T a torus, N a nilpotent ideal, $\dim(T) = \dim(N/[N,N])$, and $\mathrm{Z}(L) = 0$. Suppose, further, that the weights of T in $N/[N,N]$ are disjoint from those in $[N,N]$ and that $\Gamma(T,N)$ has no isolated vertices. Then,*

$$\mathrm{QDer}(L) = \mathrm{ad}(L) \oplus \mathrm{C}(L).$$

If $\Gamma(T,N)$ is connected, then

$$\mathrm{QDer}(L) = \mathrm{ad}(L) \oplus (I_L).$$

Proof: Since $\mathrm{Z}(L) = 0$, and $\Gamma(T,N)$ has no isolated vertices, $L = T + N$ is complete, so that $\mathrm{Der}(L) = \mathrm{ad}(L)$ (see [11] or [12]). Let U be a subspace of N such that $N = U + [N,N]$. Then U generates N, and the weights of T on U are special, so Corollary 4.16 gives the first statement. If $\Gamma(T,N)$ is connected, then L is indecomposable and the second statement follows from Theorem 4.12. □

The condition in Corollary 4.18 that the weights in $N/[N,N]$ and $[N,N]$ are disjoint is superfluous in characteristic 0 since it is already implied by the other hypotheses.

As a typical application, we get

Corollary 4.19. *Let N be a nonabelian free-nilpotent Lie algebra, of dimension > 1, over a field of characteristic p, i.e., $N = F/F^n$ where F is free and $n > 2$. Let T be a maximal torus of $\mathrm{Der}(N)$, and let L be the semidirect sum $T + N$. If either $p = 0$, or $p \geq n$,*

$$\mathrm{QDer}(L) = \mathrm{ad}(L) \oplus (I_L).$$

□

Remark 4.20. The conclusion of Corollary 4.19 fails for $\dim(N) = 1$. In that case, L is the 2-dimensional, nonabelian Lie algebra and $\mathrm{QDer}(L) = \mathfrak{gl}(L)$ (Theorem 3.8).

5. Quasi-Centroids of Lie Algebras

5.1. **Preliminary remarks.** The centroid $\mathrm{C}(L)$ is an associative algebra and for centerless L it is commutative. As we note below, $\mathrm{QC}(L)$ is not closed under composition in general. Indeed, one of our main results is that $\mathrm{QC}(L)$ is a commutative algebra if L is centerless.

First we have

Lemma 5.1. $[\mathrm{C}(L), \mathrm{QC}(L)] \subseteq \mathrm{Hom}(L, \mathrm{Z}(L))$. *Thus, if* $\mathrm{Z}(L) = 0$, $\mathrm{C}(L)$ *centralizes* $\mathrm{QC}(L)$.

Proof: Let $f \in \mathrm{C}(L)$, $g \in \mathrm{QC}(L)$, $x, y \in L$. Then $[[f,g](x), y] = [f(g(x)), y]] - [g(f(x)), y] = f([g(x), y]) - [f(x), g(y)] = f([g(x), y]) - f([x, g(y)]) = 0$. □

Note that $\mathrm{QC}(L)$ is a Jordan algebra, using the operation, $f_1 \bullet f_2 = (f_1 f_2 + f_2 f_1)/2$ for any elements $f_1, f_2 \in \mathrm{QC}(L)$. It follows that $\mathrm{QC}(L)$ is a Lie algebra with the operation $[f_1, f_2] = f_1 f_2 - f_2 f_1$ if and only if $\mathrm{QC}(L)$ is also an associative algebra (with respect to composition). We shall show, for centerless L, that $\mathrm{QC}(L)$ is commutative and so, in particular, is closed under composition. However, it need not be closed under composition otherwise, not even for $\dim(\mathrm{Z}(L)) = 1$, as the following example shows.

Example 5.2. Let N be the Heisenberg algebra, of dimension $2n+1$. Then, with respect to the usual basis (see Section 2), $\mathrm{QC}(N)$ has the form

$$\begin{pmatrix} A & B & 0 \\ C & A^t & 0 \\ e & f & g \end{pmatrix}$$

where A, B, C are $n \times n$ matrices, with B, C skew symmetric, and e, f, g are field elements. Hence, $\mathrm{QC}(L)$ is not closed under composition.

Remark 5.3. Note by Lemma 3.1(3) that we may regard $\mathrm{QC}(L)$ as an L-submodule of $\mathrm{Hom}(L, L)$.

Lemma 5.4.

(1) *For* $x \in L$, $f \in \mathrm{QC}(L)$, $[x, f(x)] = 0$ *and so* $\mathrm{ad}(x)$ *and* $\mathrm{ad}(f(x))$ *commute.*
(2) *Let* $f \in \mathrm{QC}(L), x, y \in L$. *Then* $(x \cdot f)(y) = -(y \cdot f)(x)$.
(3) *Let* $f \in \mathrm{QC}(L)$, $x \in L$. *Then* $(\mathrm{ad}(f(x))^m = (\mathrm{ad}(x))^m \circ f^m$ *for all* $m \geq 0$.

Proof: (1) and (2) are easy verifications, while (3) follows by induction on m using (1). □

Lemma 5.5.

(1) *If* $f \in \mathrm{C}(L)$, *then* $\mathrm{Ker}(f)$ *and* $\mathrm{Im}(f)$ *are ideals in* L.

(2) *If L is indecomposable, and if $0 \neq f \in \mathrm{C}(L)$, is such that x^2 does not divide the minimal polynomial of f, then f is invertible.*
(3) *If L is indecomposable and $\mathrm{C}(L)$ consists of semisimple elements then $\mathrm{C}(L)$ is a field.*

Proof: (1) is clear. (2) follows from (1), since the minimal-polynomial hypothesis forces $L = \ker(f) \oplus \mathrm{Im}(f)$. (3) follows from (2). □

5.2. **On commutativity of** $\mathrm{QC}(\mathbf{L})$**.** We show that $\mathrm{QC}(L)$ is commutative for centerless L.

Notation. Recall that a Cartan subalgebra, H, of a Lie algebra, L, is a nilpotent subalgebra of L which equals its normalizer. Also, $L = L_0 \oplus L_1$, where L_0 is the Fitting null-component, and $L_1 = \sum_{x \in H} L_{1,x}$ is the Fitting 1-component of L with respect to $\mathrm{ad}(H)$, where $L_{1,x}$ is the Fitting 1-component of L with respect to $\mathrm{ad}(x)$ (see [9, p.39 et seq]). Further, H is a Cartan subalgebra of L if and only if H is nilpotent and is the Fitting null-component of $\mathrm{ad}(H)$ (see [9, p.57]).

Also, H acts on $\mathrm{QC}(L)$, (Remark 5.3) and we let $\mathrm{QC}(L)_0, \mathrm{QC}(L)_1$ denote the Fitting null and 1-component of $\mathrm{QC}(L)$ under this action.

First we show

Lemma 5.6. *Let H be a Cartan subalgebra of L. If $f \in \mathrm{QC}(L)$, then $f(H) \subseteq H$.*

Proof: By Lemma 5.4, (1) and (3), for all $j \geq 0$, $y \in L$,

$$(\mathrm{ad}(y))^{j+1} \circ f = (\mathrm{ad}(y))^j \circ \mathrm{ad}(f(y)) = \mathrm{ad}(f(y)) \circ (\mathrm{ad}(y))^j,$$

so that f preserves the Fitting null-component of $\mathrm{ad}(y)$ for all $y \in H$. □

Benoist states the above result for the case of semisimple Lie algebras over fields of characteristic 0 [1, p.903].

Example 5.7. It is not necessarily the case that $\mathrm{QC}(L)$ preserves L_1. A counterexample, useful for other purposes, is as follows: Let L have a basis $x_{0,\cdots,}x_5$ with

$$\begin{array}{lll} [x_0, x_1] = x_1, & [x_0, x_3] = x_3, & [x_0, x_5] = x_5, \\ [x_1, x_2] = x_5, & [x_3, x_4] = x_5, & \end{array}$$

and with other products 0. Then $\mathrm{C}(L)$ is spanned by I_L and f_1, f_2, where $f_1(x_0) = x_2$, $f_1(x_1) = -x_5$, $f_2(x_0) = x_4$, $f_2(x_3) = -x_5$, while otherwise, $f_i(x_j) = 0$. And $\mathrm{QC}(L)$ is spanned by $\mathrm{C}(L)$ and f_3, where $f_3(x_1) = -x_4$, $f_3(x_3) = x_2$ with $f_3(x_i) = 0$ for $i \neq 1, 3$. A Cartan subalgebra, H, is spanned by x_0, x_2, x_4. With respect to this Cartan subalgebra, L_1 is spanned by x_1, x_3, x_5, but $f_3(L_1) \not\subseteq L_1$.

Referring back to Remark 4.17, we note that L has a derivation d such that $d(x_2) = x_4$, $d(x_3) = x_1$, while $d(x_i) = 0$ if $i \neq 2$ or 3. Note that $f_1 \circ d \neq d \circ f_1$, so that $\mathrm{C}(L) \neq \mathrm{Z}_{\mathrm{C}(L)}(\mathrm{Der}(L))$.

In view of the next Lemma, it is also worth noting that $f_3(H) = 0$ and $f_3(L_1) \subseteq \mathrm{Z}(H)$.

Lemma 5.8. *Let H be a Cartan subalgebra of L and let $H \oplus L_1$ be the Fitting decomposition of L for the action of* $\mathrm{ad}(H)$ *($H = L_0$). Let $\mathrm{QC}(L)_0 + \mathrm{QC}(L)_1$ be the Fitting decomposition of $\mathrm{QC}(L)$ regarded as an H-module. (see Remark 5.3) Then*

(1) $\mathrm{QC}(L)_0(L_1) \subseteq L_1$.
(2) $\mathrm{QC}(L)_1(H) = 0$.
(3) $\mathrm{QC}(L)_1(L_1) \subseteq \mathrm{Z}(H)$.
(4) $(H \cdot \mathrm{QC}(L)_0)(L_1) = 0$.
(5) $L_1 \cdot \mathrm{QC}(L)_0 = 0$.

Proof: The map $\mathrm{QC}(L) \otimes L \to L$ such that $f \otimes x \mapsto f(x)$ is an L-map, while, for L-modules, M, N, one has $M_0 \otimes N_1 + M_1 \otimes N_0 \subseteq (M \otimes N)_1$. Thus (1) holds and $\mathrm{QC}(L)_1(H) \subseteq L_1$. Then, by Lemma 5.6, $\mathrm{QC}(L)_1(H) \subseteq L_1 \cap H = 0$, which is (2).
(3) If $f \in \mathrm{QC}(L)_1$, then by (2) $[H, f(L_1)] = [f(H), L_1] = 0$. Thus $f(L_1) \subseteq L_0 = H$ and also $f(L_1) \subseteq \mathrm{Z}(H)$.
(4) Using (2) of Lemma 5.4 and (1) of this lemma,

$$(H \cdot \mathrm{QC}(L)_0)(L_1) = (L_1 \cdot \mathrm{QC}(L)_0)(H) \subseteq \mathrm{QC}(L)_1(H) = 0.$$

(5) Suppose that $L_1 \cdot \mathrm{QC}(L)_0 \neq 0$. Then, by the proof of (4), $(L_1 \cdot \mathrm{QC}(L)_0)(L_1) \neq 0$ and then, noting that $L_1 \cdot \mathrm{QC}(L)_0 \subseteq QC(L)_1$, (3) implies that $[(L_1 \cdot \mathrm{QC}(L)_0)(L_1), L_1] \neq 0$. Hence there exist $x, y, z \in L_1, f \in \mathrm{QC}(L)_0$ such that $[(x \cdot f)(y), z] \neq 0$. We may assume that the ground field is algebraically closed and so, without loss of generality, that x, y, z are in weight spaces for H. Say $x \in L_\alpha, y \in L_\beta, z \in L_\gamma$ for nonzero α, β, γ. Since, by (3), $0 \neq (x \cdot f)(y) \in H$ it follows that $\alpha + \beta = 0$. However, using (2) of Lemma 5.4, and the QC-property

$$0 \neq [(x \cdot f)(y), z] = [y, (x \cdot f)(z)] = -[y, (z \cdot f)(x)] = -[(z \cdot f)(y), x],$$

from which it follows, similarly, that $\alpha + \gamma = 0$ and $\beta + \gamma = 0$, which yields the contradiction $\alpha = \beta = \gamma = 0$. ☐

Lemma 5.9. *If $\mathrm{Z}(L) = 0$ then $\mathrm{QC}(L)_0 = \mathrm{C}(L)$.*

Proof: Since $\mathrm{C}(L) = \{f \in \mathrm{Hom}(L, L) \mid L \cdot f = 0\}$, we have $\mathrm{C}(L) \subseteq \mathrm{QC}(L)_0$. We must show that $L \cdot \mathrm{QC}(L)_0 = 0$. By Lemma 5.8 (5), it suffices to show that $(H \cdot \mathrm{QC}(L)_0)(H) = 0$. But, by the quasicentroid property and Lemma 5.8 (4),

$$[(H \cdot \mathrm{QC}(L)_0)(H), L_1] = [H, (H \cdot \mathrm{QC}(L)_0)(L_1)] = 0.$$

Thus $H \cdot \mathrm{QC}(L)_0(H) \subseteq \mathrm{Z}(L) = 0$. ☐

Lemma 5.10. *If $\mathrm{Z}(L) = 0$, then $L_1 \cdot \mathrm{QC}(L)_1 \subseteq \mathrm{QC}(L)_0$.*

Proof: Suppose there exists $x \in L_1, f \in \mathrm{QC}(L)_1$ such that $x \cdot f \notin \mathrm{QC}(L)_0$. We may assume the base field algebraically closed and so, without loss of generality, assume that x and f lie in weight spaces for H. Take $x \in L_\alpha$, $f \in \mathrm{QC}(L)_\beta$ for nonzero α, β then, since $x \cdot f \notin \mathrm{QC}(L)_0$, we have $\alpha + \beta \neq 0$. Thus, since $(x \cdot f)(H) = 0$ by Lemma 5.8(2), there exists $y \in L_\gamma$, with $\gamma \neq 0$ such that $(x \cdot f)(y) \neq 0$. By Lemma 5.8(3), $\alpha + \beta + \gamma = 0$. Also, there exists $z \in L_\delta$, with $\delta \neq 0$, such that $[(x \cdot f)(y), z] \neq 0$ or else, by Lemma 5.8(3), $(x \cdot f)(y) \in \mathrm{Z}(L) = 0$. By Lemma 5.4(2) and the QC property,

$$0 \neq [(x \cdot f)(y), z] = [y, (x \cdot f)(z)] = -[y, (z \cdot f)(x)] = -[(z \cdot f)(y), x].$$

From $(x \cdot f)(z) \neq 0$, we conclude $\alpha + \beta + \delta = 0$, so that $\gamma = \delta$. Now $(z \cdot f) \in \mathrm{QC}(L_1)$, else $\delta + \beta = 0$, which would imply $\alpha = 0$. Then $(z \cdot f)(y) \neq 0$ yields $\delta + \gamma + \beta = 0$ by Lemma 5.8(3). Hence $\alpha = \gamma = \delta$ and $\beta = -2\alpha$. Since $\alpha + \beta \neq 0, f(y) = f(z) = 0$, again, by Lemma 5.8(3). Then $(x \cdot f)(y) = [x, f(y)] - f([x, y]) = -f([x, y])$ so

$$0 \neq [(x \cdot f)(y), z] = -[f([x, y]), z] = -[[x, y], f(z)] = 0,$$

which is a contradiction. □

Theorem 5.11. *If L is a Lie algebra with $\mathrm{Z}(L) = 0$, then $\mathrm{QC}(L)$ is a commutative, associative algebra.*

Proof: We may assume the base field algebraically closed. Let $\mathrm{QC}(L)_0$ and $\mathrm{QC}(L)_1$ be as above. By Lemma 5.9 and Proposition 5.1, we have $[\mathrm{QC}(L)_0, \mathrm{QC}(L)] = [\mathrm{C}(L), \mathrm{QC}(L)] = 0$. Now it follows from Lemma 5.8, (2) and (3), that $\mathrm{QC}(L)_1 \circ \mathrm{QC}(L)_1 = 0$, so that $[\mathrm{QC}(L)_1, \mathrm{QC}(L)_1] = 0$. □

We have actually proved more about the structure of $\mathrm{QC}(L)$.

Theorem 5.12. *If L is a Lie algebra with $\mathrm{Z}(L) = 0$, then $\mathrm{QC}(L) = \mathrm{C}(L) \oplus A$, with $C(L) \circ A \subseteq A$ and $A \circ A = 0$.*

Proof: In the above discussion, $A = \mathrm{QC}(L_1)$. That $A \circ A = 0$ is shown in the proof of Theorem 5.11. □

5.3. **The quasicentroid, the radical, and the maximal nilpotent ideal.** This section is concerned, primarily, with the invariance of classically important ideals of L under the action of $\mathrm{QC}(L)$. Example 5.7 shows that the derived algebra of L need not be invariant under $\mathrm{QC}(L)$, even if L is centerless. However, we show that $\mathrm{Rad}(L)$ and $\mathrm{NilRad}(L)$ are invariant under $\mathrm{QC}(L)$, and further, that the image of a nilpotent QC is always contained in the nilradical.

Theorem 5.13. $\mathrm{QC}(L)$ *preserves* $\mathrm{NilRad}(L)$.

Proof: Take $f \in \mathrm{QC}(L)$, $x \in \mathrm{NilRad}(L)$. Then $f(x) \in \mathrm{Tr}(L, \mathrm{NilRad}(L))$, so that $K = (f(x)) + \mathrm{NilRad}(L)$ is an ideal in L. Since $\mathrm{ad}(x)$ is nilpotent, it follows from (3) of Lemma 5.4, that $\mathrm{ad}(f(x))$ is nilpotent, which implies that K is nilpotent. Hence $K = \mathrm{NilRad}(L)$ whence $f(x) \in \mathrm{NilRad}(L)$. □

The invariance of the radical depends on the following easy observation.

Lemma 5.14. *If L is a Lie algebra then* $\mathrm{Tr}(L, \mathrm{Rad}(L)) = \mathrm{Rad}(L)$.

Proof: $[\mathrm{Tr}(L, \mathrm{Rad}(L)), \mathrm{Tr}(L, \mathrm{Rad}(L))] \subseteq \mathrm{Rad}(L)$. □

This yields a result about a class of generalized derivations slightly larger than $\mathrm{QC}(L)$.

Lemma 5.15. *Let L be a Lie algebra. If $(f, f', 0) \in \Delta(L)$, then*

$$f(\mathrm{Rad}(L)) \subseteq \mathrm{Rad}(L).$$

Proof: Since $[f(\mathrm{Rad}(L)), L] = [\mathrm{Rad}(L), f'(L)] \subseteq \mathrm{Rad}(L)$, $f(\mathrm{Rad}(L)) \subseteq \mathrm{Tr}(L, \mathrm{Rad}(L)) = \mathrm{Rad}(L)$. □

Our investigation of the image of nilpotent QCs is motivated by the following observation about the centroid:

Lemma 5.16. *Let L be a Lie algebra, $f \in \mathrm{C}(L)$ such that $f^m = 0$ with $m \geq 0$. Then $f(L)$ is an ideal of L and satisfies $f(L)^m = 0$. In particular, $f(L) \subseteq \mathrm{NilRad}(L)$.*

Proof: As noted in Lemma 5.5, $f(L)$ is an ideal. The centroidal property of f also implies $f(L)^m = f^m(L^m) = 0$. □

Example 5.17. It is easy to construct centroidal elements of arbitrary nilpotency index m even in centerless algebras: Let L be a Lie algebra over κ with $\mathrm{Z}(L) = 0$. For any $m \geq 0$, let $L\{m\} := L[\kappa[t]/(t^m)]$ $(= L \otimes \kappa[t]/(t^m))$ where t is an indeterminant. Then $I \otimes$ (multiplication by t) is an element of $\mathrm{C}(L\{m\})$ which is nilpotent of index m. This construction with L simple is a mainstay of [1].

The image of $f \in \mathrm{QC}(L)$ need not be an ideal (see f_3 in Example 5.7). However, we still have

Theorem 5.18. *Let L be a Lie algebra, f a nilpotent element of* $\mathrm{QC}(L)$. *Then $\langle f(L) \rangle$ is nilpotent, and so $f(L) \subseteq \mathrm{NilRad}(L)$.*

For the proof of Theorem 5.18, we will need only the case $m = 2$ of the following theorem. (We remind the reader of our overall hypothesis that characteristic $\neq 2$.) However, the full statement of Theorem 5.19 presents an interesting analogue to Lemma 5.16.

Theorem 5.19. *Let L be a Lie algebra over a field, κ, of characteristic p. Suppose $f \in \mathrm{QC}(L)$ satisfies $f^m = 0$ with $m > 0$. If either $p = 0$, or $0 < m < p$, then $\langle f(L) \rangle^m = 0$.*

Notation. We represent by $[x_1, x_2, \ldots, x_n]$ the left-associated monomial

$$[[\cdots[[x_1, x_2], x_3], \ldots], x_n].$$

Proof: We need only show that, for all n, and $x_i \in L$, $1 \leq i \leq n$, $[x_1, x_2, \ldots, x_n] = 0$, whenever $x_i \in f(L)$ for at least m indices i. In turn, it suffices to show that $[f^{e_1}(x_1), f^{e_2}(x_2), \ldots, f^{e_n}(x_n)] = 0$ if $\sum_{i=1}^{n} e_i \geq m$. Further, since $[f^{e_1}(x_1), f^{e_2}(x_2)] = [x_1, f^{e_1+e_2}(x_2)]$, we may assume $e_1 = 0$. Thus, the goal is to prove, for $\sum_{i=2}^{n} e_i \geq m$,

$$[x_1, f^{e_2}(x_2), \ldots, f^{e_n}(x_n)] = 0. \tag{5.1}$$

We prove (5.1) by induction on n.

If $e_2 \geq m$, then $[x_1, f^{e_2}(x_2)] = 0$; so (5.1) holds for $n = 2$.

Let $n > 2$ and suppose (5.1) holds for products of length $< n$.

For convenience in what follows, we abbreviate the sequence $f^{e_4}(x_4), \ldots, f^{e_n}(x_n)$ by Φ; these $n-3$ factors will never change in our manipulations. We also employ $\{\mathcal{I}\}$, $\{\mathcal{J}\}$, $\{\mathcal{S}\}$, $\{\mathcal{Q}\}$, and $\{N\}$ respectively, to indicate that an argument used the induction hypothesis, the Jacobi identity, skew-commutativity, the QC relation, or equation (N), respectively.

The case $e_3 = 0$ will be handled first. We claim that, for $e_2 + \sum_{i=4}^{n} e_i \geq m$,

$$[x_1, f^{e_2}(x_2), x_3, \Phi] = 0. \tag{5.2}$$

Proof of (5.2). We have

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= [x_3, f^{e_2}(x_2), x_1, \Phi] + [[x_1, x_3], f^{e_2}(x_2), \Phi] && \{\mathcal{J}\} \\ &= [x_3, f^{e_2}(x_2), x_1, \Phi], && \{\mathcal{I}\} \end{aligned}$$

and

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= [f^{e_2}(x_1), x_2, x_3, \Phi] && \{\mathcal{Q}\} \\ &= -[x_2, f^{e_2}(x_1), x_3, \Phi]. && \{\mathcal{S}\} \end{aligned}$$

Applying these two relations,

$$\begin{aligned} [x_1, f^{e_2}(x_2), x_3, \Phi] &= -[x_2, f^{e_2}(x_1), x_3, \Phi] \\ &= -[x_3, f^{e_2}(x_1), x_2, \Phi] \\ &= [x_1, f^{e_2}(x_3), x_2, \Phi] \\ &= [x_2, f^{e_2}(x_3), x_1, \Phi] \\ &= -[x_3, f^{e_2}(x_2), x_1, \Phi] \\ &= -[x_1, f^{e_2}(x_2), x_3, \Phi] \end{aligned}$$

from which (5.2) follows.

We always have

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_2, f^{e_2}(x_1), f^{e_3}(x_3), \Phi]. \qquad \{\mathcal{Q}\}, \{\mathcal{S}\} \tag{5.3}$$

Next we claim, for $\sum_{i=2}^{n} e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi]. \tag{5.4}$$

Proof of (5.4):

$$\begin{aligned}
&[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \\
&\quad = [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + [f^{e_3}(x_3), f^{e_2}(x_2), x_1, \Phi] && \{\mathcal{J}\}, \{\mathcal{S}\} \\
&\quad = [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + [x_3, f^{e_2+e_3}(x_2), x_1, \Phi] && \{\mathcal{Q}\} \\
&\quad = [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] + 0. && \{5.2\}
\end{aligned}$$

And, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi]. \tag{5.5}$$

Proof of (5.5):

$$\begin{aligned}
[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] &= [x_1, f^{e_3}(x_3), f^{e_2}(x_2), \Phi] && \{5.4\} \\
&= -[x_3, f^{e_3}(x_1), f^{e_2}(x_2), \Phi] && \{5.3\} \\
&= -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi]. && \{5.4\}
\end{aligned}$$

And, for $\sum_{i=2}^n e_i \geq m$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi]. \tag{5.6}$$

Proof of (5.6):

$$\begin{aligned}
[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] &= -[x_3, f^{e_2}(x_2), f^{e_3}(x_1), \Phi] && \{5.5\} \\
&= [x_2, f^{e_2}(x_3), f^{e_3}(x_1), \Phi] && \{5.3\} \\
&= -[x_1, f^{e_2}(x_3), f^{e_3}(x_2), \Phi] && \{5.5\} \\
&= -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi]. && \{5.4\}
\end{aligned}$$

Also, for $\sum_{i=2}^n e_i \geq m$ with $e_2 + e_3 \geq 1$,

$$[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi]. \tag{5.7}$$

Proof of (5.7). Proof is by a (sub)induction on e_2.

When $e_2 = 0$, we have, $[x_1, x_2, f^{e_3}(x_3), \Phi] = [[x_1, x_2], f^{e_3}(x_3), \Phi] = 0$ by the main induction (on n).

Assume $e_2 > 0$ and that the corresponding result holds for $e_2 - 1$. We have,

$$\begin{aligned}
&[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] \\
&\quad = [f(x_1), f^{e_2-1}(x_2), f^{e_3}(x_3), \Phi] && \{\mathcal{Q}\} \\
&\quad = [f(x_1), f^{e_3}(x_3), f^{e_2-1}(x_2), \Phi] \\
&\qquad\qquad + [f^{e_3}(x_3), f^{e_2-1}(x_2), f(x_1), \Phi] && \{\mathcal{J}\}, \{\mathcal{S}\} \\
&\quad = [x_1, f^{e_3+1}(x_3), f^{e_2-1}(x_2), \Phi] \\
&\qquad\qquad + [x_3, f^{e_2+e_3-1}(x_2), f(x_1), \Phi] && \{\mathcal{Q}\} \\
&\quad = [x_1, f^{e_2-1}(x_2), f^{e_3+1}(x_3), \Phi] \\
&\qquad\qquad - [x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] && \{5.4\}, \{5.5\} \\
&\quad = -(e_2 - 1)[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] \\
&\qquad\qquad - [x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] && \{\mathcal{I}\} \\
&\quad = -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi].
\end{aligned}$$

Hence (5.7) holds.

Finally, for $e_2 + e_3 \geq 1$,

$$\begin{aligned} -e_2[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi] &= [x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] && \{5.7\} \\ &= -[x_1, f^{e_3}(x_2), f^{e_2}(x_3), \Phi] && \{5.6\} \\ &= e_3[x_1, f^{e_2+e_3-1}(x_2), f(x_3), \Phi]. && \{5.7\} \end{aligned}$$

Since either the characteristic, p, is 0 or $e_2 + e_3 \leq m < p$, this yields $[x_1, f^{e_2}(x_2), f^{e_3}(x_3), \Phi] = 0$, completing the proof of Theorem 5.1. ☐

Proof of Theorem 5.18: Observing that $f(\mathrm{Z}(L)) \subseteq \mathrm{Z}(L)$, we see that f naturally induces $\bar{f} \in \mathrm{QC}(L/\mathrm{Z}(L))$ wherein $\bar{f}(x + \mathrm{Z}(L)) = f(x) + \mathrm{Z}(L)$. Since $\mathrm{NilRad}(L)/\mathrm{Z}(L) = \mathrm{NilRad}(L/\mathrm{Z}(L))$, $f(L) \subseteq \mathrm{NilRad}(L)$ if and only if $\bar{f}(L/\mathrm{Z}(L)) \subseteq \mathrm{NilRad}(L/\mathrm{Z}(L))$. Thus, we may assume that $\mathrm{Z}(L) = 0$.

We have $f = f_0 + f_1$, with $f_0 \in \mathrm{C}(L)$, $f_1 \in A$, as in Theorem 5.12. Say $f^m = 0$. Then $0 = (f_0 + f_1)^m = f_0^m + m f_0^{m-1} f_1$. Since $f_0^m \in \mathrm{C}(L)$ and $m f_0^{m-1} f_1 \in A$, $f_0^m = 0$. By Lemma 5.16, $f_0(L) \subseteq \mathrm{NilRad}(L)$. Since $f_1^2 = 0$, $f_1(L) \subseteq \mathrm{NilRad}(L)$ by Theorem 5.19. ☐

It follows immediately that

Corollary 5.20. *Semisimple Lie algebras have no non-zero nilpotent* QC*s.* ☐

Corollary 5.20 also follows from the "$\mathrm{QC}(L) = \mathrm{C}(L)$" theory of the next subsection and results of Melville [14].

5.4. Relations between QC(L) and C(L) for centerless Lie algebras.

Lemma 5.21. *Let* $\mathrm{Z}(L) = 0$ *and* $f \in \mathrm{QC}(L)$ *and suppose* x^2 *does not divide the minimal polynomial of* f*. Then* $L = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$*, a direct sum of ideals.*

Proof: The hypothesis on the minimal polynomial implies at least a vector direct sum $L = \mathrm{Ker}(f) \oplus \mathrm{Im}(f)$. Also, $[\mathrm{Ker}(f), f(L)] = [f(\mathrm{Ker}(f)), L] = 0$. Since $\mathrm{Z}_L(\mathrm{Im}(L)) \cap \mathrm{Z}_L(\mathrm{Ker}(L)) = \mathrm{Z}(L) = 0$, we must have $\mathrm{Ker}(f) = \mathrm{Z}_L(\mathrm{Im}(L))$ and $\mathrm{Im}(f) = \mathrm{Z}_L(\mathrm{Ker}(L))$. If follows that that $\mathrm{Im}(L)$ and $\mathrm{Ker}(L)$ are ideals. ☐

Corollary 5.22. *If* $f \in \mathrm{QC}(L)$ *is semisimple, and* L *is centerless, then* $f \in \mathrm{Z}_{\mathrm{C}(L)}(\mathrm{GenDer}(L))$.

Proof: We may assume the base field is algebraically closed and, by Lemma 3.4, that L is directly indecomposable. Let λ be a characteristic root of f. By Lemma 5.21, $\mathrm{Ker}(f - \lambda I_L) = L$ (since $\mathrm{Ker}(f - \lambda I_L) \neq 0$). That is, $f = \lambda I_L$. ☐

Our first instance of equality in (1.6) is given by

Theorem 5.23. *If* L *is a centerless Lie algebra for which* $\mathrm{C}(L) = (I_L)$*, then* $\mathrm{QC}(L) = (I_L)$.

Proof: We may assume that the base field accommodates a Cartan subalgebra for L. By Lemma 5.9, it suffices to show that $\mathrm{QC}(L)_1 = 0$. (See Lemma 5.8 for notation.)

Claim: $L_1 \cdot \mathrm{QC}(L)_1 = 0$. Suppose the contrary. Then, by Lemmas 5.9 and 5.10, there exist $x \in L_1$ and $f \in \mathrm{QC}(L)_1$, such that $x \cdot f = I_L$. By Lemma 5.8, $f(H) = 0$ and $f(L_1) \subseteq H$, and so, by Theorem 5.19, $\langle f(L) \rangle$ is abelian. Then,

$$H = I_L(H) = (x \cdot f)(H) \subseteq [x, f(H)] + f(L) = f(L),$$

but a Cartan subalgebra of a nonabelian algebra cannot be contained in an abelian ideal, proving the claim.

Now, by Lemma 5.8, Lemma 5.4(2), and the Claim,

$$\mathrm{QC}(L)_1(L) = \mathrm{QC}(L)_1(L_1) = (H \cdot \mathrm{QC}(L)_1)(L_1) = (L_1 \cdot \mathrm{QC}(L)_1)(H) = 0.$$

□

Note that $C(L) = (I_L)$ means that the centroid coincides with the base field; algebras with this property are called *central* [9, p. 291].

If all of the elements of the centroid are semisimple, we move to the algebraic closure of the base field and the centroid may be diagonalized. Then L is a direct sum of weight spaces for the centroid, each of which is an ideal satisfying the hypothesis of Theorem 5.23. So we have,

Corollary 5.24. *If L is a centerless Lie algebra over a perfect field such that every element of $\mathrm{C}(L)$ is semisimple, then $\mathrm{QC}(L) = \mathrm{C}(L)$.* □

The next lemma includes a key to proving equality in inclusion (1.4) for centerless Lie algebras.

Lemma 5.25. *Suppose $(f, -f, 0), (f, f, f') \in \Delta(L)$. Then $\forall x, y, z \in L$,*

$$[x, f([y, z])] = [x, [f(y), z]] = [x, [y, f(z)]],$$
$$f'([x, [y, z]]) = [x, f'([y, z])].$$

Proof: For convenience, let $f' = 2g$. As in the proof of Theorem 5.27, we have

$$g([x, y]) = [f(x), y] = [x, f(y)]. \tag{5.8}$$

Observe first that $\forall\, x, y, z \in L$,

$$\begin{aligned} [g([x, y]), z] &= [[f(x), y], z] \\ &= [[f(x), z], y] + [f(x), [y, z]] \\ &= [g([x, z]), y] + [x, f([y, z])], \end{aligned}$$

so that $\forall\, x, y, z \in L$,

$$[g([x, y]), z] + [g([z, x]), y] = [x, f([y, z])] = g([x, [y, z]]). \tag{5.9}$$

Adding the three equations obtained from (5.9) by cyclically permuting x, y, z, and using (5.9), we get $\forall\, x, y, z \in L$

$$\begin{aligned} 2([g([x,y]),z] + [g([z,x]),y] + [g([y,z]),x]) \\ = \quad g([x,[y,z]] + [y,[z,x]] + [z,[x,y]]) = 0. \end{aligned}$$

Thus, $\forall\, x, y, z \in L$,

$$[g([x,y]),z] + [g([z,x]),y] = -[g([y,z]),x] = [x, g([y,z])]. \tag{5.10}$$

Comparing (5.10) with (5.9), we have $\forall\, x, y, z \in L$,

$$[x, f([y,z])] = [x, g([y,z])].$$

Using (5.8),

$$[x, f([y,z])] = [x, [f(y), z]] = [x, [y, f(z)]],$$

and

$$f'([x,[y,z]]) = 2g([x,[y,z]]) = 2[x, f([y,z])] = 2[x, g([y,z])] = [x, f'([y,z])].$$

□

Thus we have

Proposition 5.26. *Suppose* $(f, -f, 0), (f, f, f') \in \Delta(L)$.
(1) *If* $\mathrm{Z}(L) = 0$ *then* $f \in \mathrm{C}(L)$.
(2) *If* $L = [L, L]$ *then* $f' \in \mathrm{C}(L)$. □

As an immediate consequence of Proposition 5.26(1), we get

Theorem 5.27. *If* $\mathrm{Z}(L) = 0$ *then* $\mathrm{QC}(L) \cap \mathrm{QDer}(L) = \mathrm{C}(L)$. □

Theorem 5.28. *If* $\mathrm{Z}(L) = 0$ *and* $L = [L, L]$, *then* $\mathrm{QC}(L) = \mathrm{C}(L)$.

Proof: We may assume the base field sufficiently large for L to admit a Cartan subalgebra H.

Considering the action of L on $\mathrm{QC}(L)$, let $N := \mathrm{Tr}(\mathrm{QC}(L), \mathrm{C}(L))$. By Lemmas 5.9 and 5.10, $L_1 \subseteq N$. Hence $L/N = (H+N)/N$ is nilpotent. Then, since $[L/N, L/N] = L/N$, $L = N$. Thus, $\mathrm{QC}(L)_1 = H \cdot \mathrm{QC}(L)_1 \subseteq \mathrm{C}(L) = \mathrm{QC}(L)_0$, whence $\mathrm{QC}(L)_1 = 0$. □

Definition. We say that a subalgebra J of L is *taut* if the intersection of J with any abelian ideal of L is 0.

Conditions that ensure tautness include any of the following:
(1) $J \cap \mathrm{NilRad}(L) = 0$.
(2) $0 \neq x \in J \Rightarrow (\mathrm{ad}(x))^2 \neq 0$.
(3) J is a torus acting faithfully on L.

Note also, if L has a taut Cartan subalgebra then $\mathrm{Z}(L) = 0$.

Theorem 5.29. *Let L be a Lie algebra with a taut Cartan subalgebra. Then* $\mathrm{QC}(L) = \mathrm{C}(L)$.

Proof: Let H be a taut Cartan subalgebra of L. Suppose $\mathrm{QC}(L) \neq \mathrm{C}(L)$, then there exists $f \neq 0$ in $\mathrm{QC}(L)_1$. Then $f(L) \subseteq H$ and $f(H) = 0$ by (2) and (3) of Lemma 5.8, so that $f^2 = 0$. By Theorem 5.19, $\langle f(L) \rangle$ is abelian, so that H has a nonzero intersection with an abelian ideal, contradicting the tautness of H. □

Lemma 5.30. *Let L be a Lie algebra with a taut Cartan subalgebra. If $0 \neq f \in \mathrm{QC}(L) = \mathrm{C}(L)$, then f cannot be nilpotent. In fact, the minimal polynomial of f cannot be divisible by x^2.*

Proof: Let H be a taut Cartan subalgebra of L. Suppose $f \in \mathrm{QC}(L)$ has minimum polynomial $x^2 P(x)$. Letting $g = fP(f) \in \mathrm{C}(L)$, we have $g^2 = 0$. By Lemma 5.16, $g(L)$ is an abelian ideal. Since $g(H) \subseteq H$ (Lemma 5.10), $g(H) = 0$. Then $g([H, L]) = [g(H), L] = 0$. However $L = H + [L, H]$, so this implies $g = 0$, which is a contradiction. □

Corollary 5.31. *Let L be a directly indecomposable Lie algebra over a perfect field with a taut Cartan subalgebra. Then $\mathrm{QC}(L)$ is a field in $\mathrm{Z}(\mathrm{GenDer}(L))$.*

Proof: By Lemma 5.30, $\mathrm{QC}(L)$ consists of semisimple elements. So, by Theorem 5.29 (or by Corollary 5.22), $\mathrm{QC}(L) = \mathrm{C}(L)$. By Lemma 5.5(3), $\mathrm{C}(L)$ is a field. That $\mathrm{C}(L)$ centralizes $\mathrm{GenDer}(L)$ follows from Corollary 5.22. □

A torus as in Section 4.2 (i.e., $L = T_0$, and $\mathrm{Z}(L) = 0$) is a taut Cartan subalgebra. This, in addition to Proposition 3.3(1), allows us to bring the above results together with those of Section 4. For example, using Corollary 4.16 we have:

Theorem 5.32. *Let L contain a torus, T, with $T = L_0$. Suppose that $\mathrm{Z}(L) = 0$, and that L is generated by special weight spaces. Then*

$$\mathrm{GenDer}(L) \;=\; \mathrm{Der}(L) \oplus \mathrm{C}(L),$$

a direct sum as Lie algebras. □

and:

Theorem 5.33. *If L is a parabolic subalgebra of a split simple Lie algebra of rank > 1 over a field κ of characteristic 0, then*

$$\mathrm{GenDer}(L) \;=\; \mathrm{ad}(L) \oplus (I_L) \;\simeq\; L \oplus \kappa. \qquad \square$$

6. Invariance of the Radical under Generalized Derivations.

We prove the result in the section title for Lie algebras in characteristic 0. We shall need the following lemmas.

Lemma 6.1. *Let L be a semisimple Lie algebra over a field of characteristic zero. If $(f, f, 0) \in \Delta(L)$, then $f = 0$.*

Proof: We may assume the algebra simple and the ground field, κ, algebraically closed. If $\mathrm{rank}(L) > 1$, the result follows from Theorem 4.12. That $\Delta(\mathfrak{sl}_2(\kappa))$ has no nonzero elements of the form $(f, f, 0)$ is a simple computation. □

Lemma 6.2. *Let S be a subalgebra and K an ideal in L such that $L = S + K$ is a semidirect sum. Suppose $(f, f, f') \in \Delta(L)$ and $k \in K$. Define $h\colon S \to S$ by*

$$h(s) \equiv f([s,k]) - f'([s,k]) \pmod{K}$$

for $s \in S$. Then $(h, h, 0) \in \Delta(S)$.

Proof: By Lemma 3.9(1), $\delta f = (f' - f) \circ \mu \in B^2(L, L) \cap B^2(L, \mathring{L})$. Using Lemma 3.9(2), if $s, s' \in S, k \in K$,

$$[s, (f' - f)([s', k])] + [(f' - f)([s, k]), s'] \equiv 0 \pmod{K}.$$ □

Theorem 6.3. *Let L be a Lie algebra over a field of characteristic 0, and let K be an ideal in L such that L/K is simple. Then, for any $f \in \mathrm{QDer}(L)$, $f(K) \subseteq K$.*

Proof: Suppose false and let $L = S + K$, with S simple, be a counterexample. Since K is self-transporting, any QC of L preserves K. Hence, by Proposition 3.3(1), there is some $(f, f, f') \in \Delta(L)$ with $f(K) \not\subseteq K$.

Let $P = \{(f, f, f') \mid (f, f, f') \in \Delta(L)\}$. Then P is an S-module via $s \cdot (f, f, f') = (s \cdot f, s \cdot f, s \cdot f')$. Let M denote the submodule $\{(f, f, f') \in \Delta(L) \mid f(K) \subseteq K\}$. Let N be an S-module complement to M in P, so that $P = M \oplus N$. By hypothesis, $N \neq 0$, so let $(0, 0, 0) \neq (f, f, f') \in N$.

By Lemmas 6.1 and 6.2, for all $s \in S$, $k \in K$,

$$f([s,k]) \equiv f'([s,k]) \equiv [s, f(k)] \bmod (K),$$

the latter equivalence by the quasiderivation property. In particular, for all $s \in S$, $(s \cdot f)(K) \subseteq K$, so that $s \cdot (f, f, f') \in M \cap N = 0$. Thus,

$$s \cdot f = s \cdot f' = 0.$$

One, of many, useful consequences is

$$[f(s), x] = [s, (f' - f)(x)],$$

for $s \in S$, $x \in L$.

Let $\bar{f}\colon S \to S$ be determined by $\bar{f}(s) \equiv f(s) \bmod (K)$. Since S is simple, and $S \cdot f = 0$, it follows by Schur's Lemma that $\bar{f} = cI_S$ for some scalar c. Reassigning $(f, f, f') := (f - cI_L, f - cI_L, f' - 2cI_L)$, we force

$$f(S) \subseteq K,$$

while retaining the properties $S \cdot f = S \cdot f' = 0$ and $f(K) \not\subseteq K$.

We define a nonascending sequence of subspaces

$$L = K_0 \supset K_1 \supset \cdots \supset K_m \supset K_{m+1} = K_{m+2} = \cdots$$

wherein, for $i > 0$, $K_i = \{k \in K \mid f(k) \in K_{i-1}\}$ and $m+1$ is the minimal integer such that $f(K_{m+1}) \subseteq K_{m+1}$. Note that $K_1 = K$. Since $S \cdot f = 0$, K_i is an S-submodule of L. Since f induces S-injections $K_i/K_{i+1} \to K_{i-1}/K_i$, for $i \geq 1$, we see inductively (on i) that $S \cdot K_i/K_{i+1} = K_i/K_{i+1}$. This implies, in particular, that $L^S \subseteq \bigcap_i K_i$.

We claim that, for all i, j,

$$[K_i, K_j] \subseteq K_{i+j}.$$

Proof of claim: By induction on $i+j$. The statement is clear for $i+j=0$. Assume that $i+j \geq 1$ and $[K_{i'}, K_{j'}] \subseteq K_{i'+j'}$ for $0 \leq i'+j' < i+j$. We have to show $f([K_i, K_j]) \subseteq K_{i+j-1}$. Since, $f([K_i, K_j]^S) \subseteq L^S \subseteq K_{i+j-1}$, we need only show $[S, f([K_i, K_j])] \subseteq K_{i+j-1}$. We use

$$[S, f([K_i, K_j])] \subseteq [S, (f' - f)([K_i, K_j])] + [S, f'([K_i, K_j])],$$

and deal with each of the terms on the right. First,

$$[S, (f' - f)([K_i, K_j])] \subseteq [f(S), [K_i, K_j]] \subseteq [K_0, [K_i, K_j]].$$

According to whether $i = 0$, or $i > 0$, respectively, use $K_j \subseteq K_{j-1}$, or $K_i \subseteq K_{i-1}$, respectively, together with the induction hypothesis, to conclude $[K_i, K_j] \subseteq K_{i+j-1}$; but, again by the induction hypothesis, $[K_0, K_{i+j-1}] \subseteq K_{i+j-1}$. Second,

$$[S, f'([K_i, K_j])] \subseteq f'([K_i, K_j] \subseteq [f(K_i), K_j] + [K_i, f(K_j)]$$

According to whether $i = 0$, or $i > 0$, respectively, use $f(K_i) \subseteq K_0$ and $K_j \subseteq K_{j-1}$, or $f(K_i) \subseteq K_{i-1}$, respectively, together with the induction hypothesis, to conclude $[f(K_i), K_j] \subseteq K_{i+j-1}$; similarly, $[K_i, f(K_j)] \subseteq K_{i+j-1}$. Thus, the claim has been established.

One consequence of the last claim is that K_i is an ideal in L for all i. In particular, K_{m+1} is an ideal stabilized by f. Clearly, L/K_{m+1} also stands as a counterexample to the theorem, so that we may assume $K_{m+1} = 0$. Consequently, $L^S = 0$.

Next, we claim that, for $i \geq 0$ and $k \in K_i$,

$$f'(k) \equiv f(k) \bmod (K_{i+1}).$$

Proof of claim: Since, $[S, K_i] = K_i$, the observation

$$(f' - f)([s, k]) = [s, (f' - f)(k)] = [f(s), k] \in [K_1, K_i] \subseteq K_{i+1},$$

for $s \in S$, $k \in K_i$, establishes the claim.

Since $[K_i, K_j] \subseteq K_{i+j}$, for all i, j, we can make

$$\hat{L} = K_0/K_1 \;\oplus\; K_1/K_2 \;\oplus\; \cdots \;\oplus\; K_{m-1}/K_m$$

into a graded Lie algebra by a standard construction. Namely, for $x \in K_i$, $y \in K_j$,

$$[x + K_{i+1}, y + K_{j+1}] = [x, y] + K_{i+j+1}.$$

It is immediate that

$$\mathrm{Rad}(\hat{L}) = K_1/K_2 \;\oplus\; K_2/K_3 \;\oplus\; \cdots \;\oplus\; K_{m-1}/K_m.$$

It is straightforward to verify that $(\hat{f}, \hat{f}, \hat{f}') \in \Delta(\hat{L})$, where, $\hat{f}(K_0/K_1) = \hat{f}'(K_0/K_1) = 0$ and, for $i > 0$ and $k \in K_i$,

$$\hat{f}(k + K_{i+1}) = f(k) + K_i,\; \hat{f}'(k + K_{i+1}) = f'(k) + K_i.$$

However, for $k \in K_i$, $f(k) \equiv f'(k) \mod (K_{i+1})$. Hence, $\hat{f} = \hat{f}' \in \mathrm{Der}(\hat{L})$, so that $\hat{f}$ preserves the radical of $\hat{L}$. Equivalently, $f(K) \subseteq K$, contradicting our assumption about f. The theorem is proved. □

The main result of this section is

Theorem 6.4. *Let L be a Lie algebra over a field of characteristic 0 and $f \in \mathrm{GenDer}(L)$. Then f preserves $\mathrm{Rad}(L)$.*

Proof: By Lemma 5.15, the result holds for $f \in \mathrm{QC}(L)$. We may assume that $f \in \mathrm{QDer}(L)$. If L is solvable, the result holds trivially. Otherwise, let K be a maximal proper ideal of L containing the radical, and let $(f, f, f') \in \Delta(L)$. By Theorem 6.3, f preserves K. Then $f'([K, K]) \subseteq K$ and so the restriction, $f|_K$, of f to K is in $\mathrm{QDer}(K)$. We may assume, inductively (on the dimension), that $f|_K$ preserves $\mathrm{Rad}(K)$. However $\mathrm{Rad}(K) = \mathrm{Rad}(L)$. □

7. Applications.

If V is a vector space over a field, then a skew-symmetric, nonassociative algebra structure on V is an element, μ, of $\mathcal{M}(V) := \mathrm{Hom}(V \wedge V, V)$. We regard $\mathcal{M}(V)$ as an affine algebraic variety. The set of $\mu \in \mathcal{M}(V)$ satisfying the Jacobi identity is the algebraic subvariety, $\mathcal{L}(V)$, of Lie algebra structures on V. We mention three actions of $\mathrm{GL}(V)$ on $\mathcal{M}(V)$:

Action 1: $f \cdot_1 \mu = f \circ \mu \circ (f^{-1} \wedge f^{-1})$
Action 2: $f \cdot_2 \mu = \mu \circ (f^{-1} \wedge I_V)$
Action 3: $f \cdot_3 \mu = f \circ \mu$

for $f \in \mathrm{GL}(V)$ and $\mu \in \mathcal{M}(V)$.

Action 1 leaves $\mathcal{L}(V)$ fixed. Indeed, if $\gamma = f \cdot_1 \mu$, then the Lie algebra (V, γ) is isomorphic to (V, μ). If the orbit, $\mathrm{GL}(V) \cdot_1 \mu$ is a Zariski open subset of $\mathcal{L}(V)$, then (V, μ) is called *rigid* (see, e.g., [4, 16]). It is well known that if $\mathrm{H}^2(L, L) = 0$, then L is rigid (though the converse is false [17]).

Actions 2 and 3 do not always preserve $\mathcal{L}(V)$, so in these cases, the interesting questions seem to be:

(1) When is $f \cdot_i \mu \in \mathcal{L}(V)$ for $\mu \in \mathcal{L}(V)$, $f \in \mathrm{GL}(V)$, $i = 2, 3$?
(2) If $f \cdot_i \mu \in \mathcal{L}(V)$, under what conditions is $f \cdot_i \mu$ isomorphic to μ?

Action 2 will be the subject of Section 7.1, and Action 3 that of 7.2.

7.1. **Projective doubles of Lie algebras.** Given a Lie algebra (L, μ), a *projective double* on (L, μ) is a Lie algebra (L, ρ) (i.e., the vector spaces are the same) such that

$$\mathrm{ad}_\rho(L) \subseteq \mathrm{Der}(L).$$

We call a projective double *inner* if $\mathrm{ad}_\rho(L) \subseteq \mathrm{ad}_\mu(L)$, and we call $f \in \mathrm{Hom}(L, L)$ a *doubling* of (L, μ) provided (L, ρ_f) is a Lie algebra (and hence a projective double of L) where

$$\rho_f(x, y) = (f \cdot_2 \mu)(x, y) = \mu(f(x), y)$$

for all $x, y \in L$. We denote by $\mathrm{DB}(L)$ the set of doublings of L.

Any inner projective double, (L, ρ), on (L, μ) gives rise to a doubling, f, of (L, μ) such that $\rho_f = \rho$. For this, one simply chooses a basis e_i of L and defines f by choosing $f(e_i)$ so that $\mathrm{ad}_\mu(f(e_i)) = \mathrm{ad}_\rho(e_i)$. If (L, μ) is centerless, this doubling, f, is actually unique. Note further, that, since ρ_f must be skew symmetric, a doubling, f, of (L, μ) must be an element of $\mathrm{QC}(L, \mu)$ so that

$$\mathrm{C}(L) \subseteq \mathrm{DB}(L) \subseteq \mathrm{QC}(L).$$

The next result is essentially [8, Theorem 2].

Theorem 7.1. [Ikeda] *Let (L, ρ) be an inner projective double of a centerless Lie algebra, (L, μ). Then there exists a unique doubling, f, of (L, μ) such that*

(1) *f is a homomorphism of (L, ρ) into (L, μ),*
(2) *f is in the centroid of (L, ρ),*
(3) $\ker(f) = \mathrm{Z}(L, \rho)$.

Proof: Choose the unique doubling, f, as discussed above. We show that f is a homomorphism $f\colon (L, \rho) \to (L, \mu)$.

$$\begin{aligned} ad_\mu(f \circ \rho(x, y)) &= ad_\rho(\rho(x, y)) \\ &= ad_\rho(x) ad_\rho(y) - ad_\rho(y) ad_\rho(x) \\ &= ad_\mu(f(x)) ad_\mu(f(y)) - ad_\mu(f(y)) ad_\mu(f(x)) \\ &= ad_\mu(\mu(f(x), f(y))). \end{aligned}$$

Since (L, μ) is centerless, f is a homomorphism.

Thus, for $x, y \in L$,

$$f \circ \rho(x, y) = \mu(f(x), f(y)) = \rho(f(x), y)$$

which is (2).

(3) is clear since (L,μ) is centerless. □

Lemma 7.2. *Let $L = (L,\mu)$ be centerless. If $f \in \mathrm{DB}(L)$ is nonsingular then $f \in \mathrm{C}(L)$.*

Proof: $f(\mu(x,y)) = f(\mu(ff^{-1}(x),y)) = f(\rho_f(f^{-1}(x),y)) = \mu(x,f(y))$, the last equality by Theorem 7.1(1). □

Theorem 7.3. *Let L be a directly indecomposable Lie algebra with a taut Cartan subalgebra. Then every inner projective double of L is isomorphic to L.* □

Proof: By Theorem 5.29, Lemma 5.30, and Lemma 5.5(2), any nonzero element of $\mathrm{QC}(L)$ is invertible. The result follows by Lemma 7.2 and Theorem 7.1(1). □

Corollary 7.4. *If L is a simple Lie algebra, then any nonabelian inner projective double of L is isomorphic to L.* □

As another special case, we recover a result equivalent to a remark in [8, last paragraph].

Corollary 7.5. [Ikeda] *If L is a parabolic subalgebra of a simple Lie algebra in characteristic 0 then every projective double of L is isomorphic to L.*

Proof: Such algebras are complete [11, 20]. □

Corollary 7.6. *Let $L = T + N$ where T is a torus acting faithfully on N, N is a nilpotent ideal with $\dim(N/N^2) = \dim(T)$ and the weights of the T-module structure induced on N/N^2 are disjoint from the weights in $[N,N]$, and further, L is indecomposable. Then every nonabelian projective double of L is isomorphic to L.*

Proof: Such algebras are complete [11, Prop. 4.1]. □

There are natural occurrences of the situation of Corollary 7.6 (in addition to Borel subalgebras of semisimple Lie algebras in characteristic 0, which are covered by Corollary 7.5), e.g., $T+N$ with N nonabelian free nilpotent either in characteristic 0 or when the characteristic exceeds the index of nilpotency of N; see also [11, 12].

Example 7.7. It is not the case in general, for centerless, indecomposable L with $\mathrm{C}(L) = \mathrm{QC}(L)$, that every inner projective double of L is isomorphic to L. To produce a counterexample, we note first that a doubling, f yields an isomorphic projective double, (L,ρ_f) if and only if f is nonsingular. Also, by Corollary 5.22 and Lemma 5.5(2), if L is indecomposable, then a semisimple doubling is nonsingular. Thus, if L is centerless and indecomposable, then every projective double of L is isomorphic to L unless $\mathrm{C}(L)$ contains nilpotent elements. On the other hand, if $0 \neq f \in \mathrm{C}(L)$ with f nilpotent, then the projective double, (L, ρ_f), induced by f is not isomorphic to L. So, for our counterexample, it suffices to display a centerless, indecomposable Lie algebra, L, with $\mathrm{C}(L) = \mathrm{QC}(L)$ while $\mathrm{C}(L)$ contains nilpotent elements. Example 5.17 yields many such algebras but we need the following lemma.

Recall the construction of $L\{m\}$ in Example 5.17.

Lemma 7.8. *Let L be a centerless indecomposable Lie algebra over κ. Then $L\{m\}$ is centerless and indecomposable for all $m \geq 0$.*

Proof: It is clear that $L\{m\}$ is centerless. We prove the indecomposability by induction on m. For $m = 0$, we have $L\{0\} \simeq L$. Assume the lemma is true for m. Suppose $L\{m+1\} = A \oplus B$, a nontrivial direct sum of ideals. The natural homomorphism $\kappa[t]/(t^{m+1}) \to \kappa[t]/(t^m)$ induces an epimorphism $\pi\colon L\{m+1\} \to L\{m\}$. Thus $L\{m\} = \pi(A) + \pi(B)$. Since $[\pi(A) \cap \pi(B), L\{m\}] = [\pi(A) \cap \pi(B), \pi(A) + \pi(B)] \subseteq [\pi(A), \pi(B)] = 0$, we have $\pi(A) \cap \pi(B) \subseteq \mathrm{Z}(L\{m\}) = 0$. Thus, by the induction hypothesis, one of $\pi(A), \pi(B)$ is 0. Without loss of generality, $\pi(A) = 0$ and so $\pi(B) = L\{m\}$. Then $A \subseteq L \otimes (t^m)$ and $B + L \otimes (t^m) = L\{m+1\}$. But then $[L\{m+1\}, A] = [B, A] = 0$, contradicting $\mathrm{Z}(L\{m+1\}) = 0$. □

Thus, if L is centerless, indecomposable and $L = [L, L]$, then $L\{m\}$ has these properties and also, by Theorem 5.28, $\mathrm{C}(L\{m\}) = \mathrm{QC}(L\{m\})$. Now, $I \otimes (\text{multiplication by } t)$ is a nilpotent element of $\mathrm{C}(L\{m\})$.

Example 7.9. Returning to the inclusions, $\mathrm{C}(L) \subseteq \mathrm{DB}(L) \subseteq \mathrm{QC}(L)$, we observe that Example 5.7 gives a centerless, directly indecomposable Lie algebra for which both inclusions are proper:

First, $f_3 \in \mathrm{DB}(L) \setminus \mathrm{C}(L)$. (Note that (L, ρ_{f_3}) is the direct sum of the Heisenberg algebra spanned by x_1, x_3, x_5, with $[x_1, x_3] = x_5$ and a 3-dimensional abelian algebra; the homomorphism induced by f_3 maps this to a 2-dimensional abelian subalgebra of L spanned by x_2, x_4.)

To show that the second inclusion can be proper, note, by Lemma 7.2, that it suffices to produce a nonsingular $f \in \mathrm{QC}(L) \setminus \mathrm{C}(L)$, and $I + f_3$ is such a QC.

From this last example, one sees also that $\mathrm{DB}(L)$ need not be a subspace of $\mathrm{QC}(L)$: f_3 and I_L are in $\mathrm{DB}(L)$ but $I_L + f_3$ is not. However, we shall show (Corollary 7.14) that, for centerless L, $\mathrm{DB}(L)$ is closed under multiplication (composition).

Notation. Let $f \in \mathrm{QC}(L)$. Define $A_f, B_f, C_f \in \mathrm{C}^3(L,L)$ and $h_f \in \mathrm{C}^2(L,L)$ so that, for $x,y,z \in L$,

$$\begin{aligned} A_f(x,y,z) &:= [f(x),[f(y),z]] + [f(y),[f(z),x]] + [f(z),[f(x),y]], \\ B_f(x,y,z) &:= [x,[f(y),z]] + [y,[f(z),x]] + [z,[f(x),y]], \\ C_f(x,y,z) &:= [f(x),[y,z]] + [f(y),[z,x]] + [f(z),[x,y]], \\ h_f(x,y) &:= [f(x),f(y)]. \end{aligned}$$

Using the Jacobi identity and the centroid and quasicentroid properties, we see that

Lemma 7.10.

(1) *If $f \in \mathrm{C}(L)$, then $A_f = B_f = C_f = 0$.*
(2) *If $f \in \mathrm{QC}(L)$, then $B_{f^2} = -2A_f$ and $C_f = -2B_f$.* □

In light of the cohomological connections in the next subsection, it is interesting to note

Proposition 7.11. *If $f \in \mathrm{QC}(L)$ and the characteristic of the base field is either 0 or > 3, then f is a doubling if and only if $h_f \in Z^2(L,L)$.*

Proof: The Jacobi identity on the multiplication $\mu \circ (f \wedge I)$ is

$$A_f = 0,$$

while the cocycle condition applied to h_f is

$$B_{f^2} - C_{f^2} = 0, \text{ or } -6A_f = 0.$$ □

Lemma 7.12. *Let $\mathrm{Z}(L) = 0$ and $f \in \mathrm{QC}(L)_1$ (notation as in Section 5.2). Then $B_f = 0$ implies $f = 0$.*

Proof: Since $f(H) = 0$ and $f(L_1) \subseteq \mathrm{Z}(H)$, we have $0 = B_f(H, L_1, L_1) = [H,[f(L_1),L_1]]$. Since $[f(L_1),L_1] \subseteq L_1$, we have $[f(L_1),L_1] = 0$ by Lemma 5.8(3) and and therefore $f(L_1) \subseteq \mathrm{Z}(L) = 0$ by Lemma 5.8(2). □

Theorem 7.13. *Let L be a centerless Lie algebra and let $f \in \mathrm{QC}(L)$. Then $f \in \mathrm{DB}(L)$ if and only if $f^2 \in \mathrm{C}(L)$.*

Proof: We may assume the base field is large enough (e.g., algebraically closed) to assure that L has a Cartan subalgebra. Suppose $f = f_0 + f_1$ with $f_0 \in \mathrm{QC}(L)_0 = \mathrm{C}(L), f_1 \in \mathrm{QC}(L)_1$. Since $QC(L)_1^2 = 0$, $f^2 = f_0^2 + 2f_0f_1$ and so $B_{2f_0f_1} = B_{f^2} = -2A_f$. Now, $f \in \mathrm{DB}(L)$ if and only if $A_f = 0$ (proof of Proposition 7.11). But, since $f_0f_1 \in \mathrm{QC}(L)_1$, $B_{2f_0f_1} = 0$ if and only if $2f_0f_1 = 0$ (Lemma 7.12), which holds if and only if $f^2 = f_0^2 \in \mathrm{C}(L)$. □

Corollary 7.14. *If $\mathrm{Z}(L) = 0$ then $\mathrm{DB}(L)$ is closed under composition.*

Proof: By Theorem 7.13 and commutativity of $\mathrm{QC}(L)$ (Theorem 5.11). □

7.2. **Quasiderivations and robustness.** Let (L, μ) be a Lie algebra and f a nonsingular element of $\mathrm{Hom}(L, L)$. Observe that $(L, f \cdot_3) \equiv (L, f \circ \mu)$ is a Lie algebra if and only if f satisfies: $\mu(x, f(\mu(y, z))) + \mu(y, f(\mu(z, x))) + \mu(z, f(\mu(x, y))) = 0$, for all $x, y, z \in L$.

Definitions. Let (L, μ) be a Lie algebra and f a nonsingular element of $\mathrm{Hom}(L, L)$ such that $(L, f \circ \mu)$ is a Lie algebra. We call such $(L, f \circ \mu)$ a *perturbation* of (L, μ), and the perturbation is said to be *inessential* if $f \circ \mu = c \circ \mu$ for some $c \in \mathrm{C}(L)$. We say (L, μ) is *robust* if every perturbation of (L, μ) is inessential.

Proposition 7.15. *An inessential perturbation of (L, μ) is necessarily isomorphic to (L, μ).*

Proof: Suppose $f \circ \mu = c \circ \mu$ where f nonsingular and $c \in \mathrm{C}(L, \mu)$. It suffices to show that $f \circ \mu = c' \circ \mu$ with $c' \in \mathrm{C}(L, \mu)$ and c' nonsingular, for such a c' induces an isomorphism from $(L, f \circ \mu)$ to (L, μ).

Let $n := \dim(L)$ and $K := \mathrm{Ker}(c^n)$. Since $f^n(\mu(K, L)) = c^n(\mu(K, L)) = \mu(c^n(K), L) = 0$, $K \subset \mathrm{Z}(L, \mu)$ by the nonsingularity of f. The nonsingularity of f also gives $K \cap \mu(L, L) = 0$. Thus, we may choose $M \supset \mu(L, L)$ so that $L = K \oplus M$ (this is a direct sum of ideals for (L, μ)). With respect to this decomposition, let π be the projection of L to M. Define $c' \in \mathrm{Hom}(L, L)$ so that $c'(m) = \pi(c(m))$ for $m \in M$ and $c'(k) = k$ for $k \in K$. Since c induces a nonsingular transformation $L/K \to L/K$, c' is a bijection. Since $c'(x) = c(x)$ for $x \in \mu(L, L)$ and $c'(x) \equiv c(x) \pmod{\mathrm{Z}(L, \mu)}$ for all $x \in L$, we have $c' \in \mathrm{C}(L, \mu)$. □

Following [18], we define $\mathrm{sq}^2 \colon \mathrm{C}^2(L, L) \to \mathrm{C}^3(L, L)$ by

$$\mathrm{sq}^2(\nu)(x, y, z) = \nu(x, \nu(y, z)) + \nu(y, \nu(z, x)) + \nu(z, \nu(x, y))$$

so that (L, ν) is a Lie algebra if and only if $\mathrm{sq}^2(\nu) = 0$. If (L, μ) is a Lie algebra, then for $h \in \mathrm{Hom}(L, L)$,

$$h \circ (\delta(h \circ \mu)) = \mathrm{sq}^2(h \circ \mu),$$

where $\delta \colon \mathrm{C}^2(L, L) \to \mathrm{C}^3(L, L)$ is the coboundary map (computed with respect to $L = (L, \mu)$). Hence, if h is nonsingular, then $(L, h \circ \mu)$ is a Lie algebra if and only if $h \circ \mu \in \mathrm{Z}^2(L, L)$. This leads to the following.

Proposition 7.16. *Let $L = (L, \mu)$ be a Lie algebra over a field κ, with $|\kappa| > \dim(L)$. Then L is robust if and only if $\mathrm{Z}^2(L, L) \cap \mathrm{B}^2(L, \mathring{L}) = \mathrm{C}(L) \circ \mu$.*

Proof: The sufficiency follows immediately from the above. Conversely, suppose (L, μ) is robust and $h \circ \mu \in Z^2(L, L)$, then, using the field hypothesis, we may choose $c \in \kappa$ such that $cI_L + h$ is nonsingular. Since $(cI_L + h) \circ \mu \in \mathrm{Z}^2(L, L)$, $(cI_L + h) \circ \mu \in \mathrm{C}(L) \circ \mu$ by robustness. Hence also $h \circ \mu \in \mathrm{C}(L) \circ \mu$. □

Thus, it is worth highlighting a cohomological condition that characterizes the collapsing of the second inclusion in (1.3).

Proposition 7.17. $\mathrm{QDer}(L) = \mathrm{Der}(L) + \mathrm{C}(L)$ *if and only if* $\mathrm{B}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L}) = \mathrm{C}(L) \circ \mu$.

Proof: Suppose $\mathrm{B}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L}) = \mathrm{C}(L) \circ \mu$. If $f \in \mathrm{QDer}(L)$ then, by Lemma 3.9(1), $\delta f = \mathring{\delta}(f'-f) \in \mathrm{B}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L})$, so that $(f'-f)\circ\mu = g\circ\mu$ with $g \in \mathrm{C}(L)$. Then $f - g \in \mathrm{Der}(L)$.

Conversely, suppose $\mathrm{QDer}(L) = \mathrm{Der}(L) + \mathrm{C}(L)$. If, for $f, g \in \mathrm{Hom}(L,L)$, $\delta f = \mathring{\delta} g \in \mathrm{B}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L})$, then $(f,f,f+g) \in \Delta(L)$ by Lemma 3.9(1). Hence, $f = d + c$ with $d \in \mathrm{Der}(L)$, $c \in \mathrm{C}(L)$, so that $\delta f = c \circ \mu$. □

Corollary 7.18. *If* $\mathrm{H}^2(L,L) = 0$ *and if* $\mathrm{QDer}(L) = \mathrm{Der}(L) + \mathrm{C}(L)$, *then* L *is robust.* □

As a consequence we have

Theorem 7.19. *If* (L,μ) *is a parabolic subalgebra of a split simple Lie algebra, of rank* > 1, *over a field of characteristic* 0, *then* L *is robust.* □

The cohomological condition in Corollary 7.18 indicates the existence of many algebras that are both robust and rigid (see also [11]). It is useful to illustrate the independence of these properties with the following examples.

Example 7.20. The 3-dimensional simple Lie algebra is an example of a rigid ($\mathrm{H}^2(L,L) = 0$) but non-robust Lie algebra. It is indecomposable, which is of interest because decomposables are trivially non-robust. The non-robustness is easy to see by looking at the "non-split form" over the complexes, i.e., the "i-j-k" vectors. Any diagonal h makes $(L, h\circ\mu)$ into a Lie algebra. Indeed, any symmetric matrix works.

Another example is given by the 2-dimensional nonabelian Lie algebra.

Example 7.21. A robust but non-rigid Lie algebra; a Lie algebra, L, for which $\mathrm{H}^2(L,L) \neq 0$ and yet $\mathrm{Z}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L}) = \mathrm{B}^2(L,L) \cap \mathrm{B}^2(L,\mathring{L}) = (\mu)$. The example is of the form $T+N$ where T is a 3-dimensional torus and N is 7-dimensional with 1-dimensional weight spaces, the weights being (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1). Let x_1, x_2, x_3 be the generators of N corresponding to the simple weights and so that N has basis $\{x_1, x_2, x_3, [x_1,x_2], [x_2,x_3], [x_1,x_3], [x_1,[x_2,x_3]]\}$. We form the multiplication with $[x_1,x_2]$ central, so that

$$[x_2,[x_1,x_3]] = [x_1,[x_2,x_3]],\ [x_3,[x_1,x_2]] = 0.$$

The non-rigidity of L is demonstrated by the 1-parameter family of multiplications with the same basis and T-weight system, with multiplication in N satisfying

$$[x_2,[x_1,x_3]] = a[x_1,[x_2,x_3]],\ [x_3,[x_1,x_2]] = (a-1)[x_1,[x_2,x_3]],$$

where $a \in$ the base field (which we assume to be infinite). The members of the family with $a \neq 0,1$ are not isomorphic to L (e.g., for $a \neq 0,1$, $\mathrm{Z}(N) = ([x_1,[x_2,x_3]])$).

Example 7.22. Finally, a Lie algebra satisfying $\mathrm{B}^2(L,L)\cap\mathrm{B}^2(L,\mathring{L})=(\mu)$ but $\mathrm{Z}^2(L,L)\cap\mathrm{B}^2(L,\mathring{L})\neq(\mu)$ is given by

$$\begin{array}{lll}[x_1,x_3]=x_3, & [x_1,x_5]=x_5, & [x_1,x_6]=x_6,\\ [x_1,x_7]=2x_7, & [x_2,x_4]=x_4, & [x_2,x_5]=x_5,\\ [x_2,x_6]=2x_6, & [x_2,x_7]=x_7, & [x_3,x_4]=x_5.\end{array}$$

The verification is facilitated by the following: that $\mathrm{QDer}(L)=\mathrm{Der}(L)+\mathrm{C}(L)$ follows from Theorem 4.12 (the torus is spanned by x_1,x_2), hence, by Proposition 7.17, $\mathrm{B}^2(L,L)\cap\mathrm{B}^2(L,\mathring{L})=(\mu)$, ($\mu$ is $[,]$); on the other hand, $(L,h\circ\mu)$ is a Lie algebra where $h(x_i)=x_i$, for $1\leq i\leq 5$, and $h(x_6)=2x_6$, $h(x_7)=3x_7$, so that $\mathrm{Z}^2(L,L)\cap\mathrm{B}^2(L,\mathring{L})\neq(\mu)$ by Proposition 7.16 Corollary 7.18.

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