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Hamiltonicity of Amalgams

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Abstract. An amalgam is obtained from two Halin graphs having skirting cy
same length. We are interested in hamiltonicity of amalgams constructed from tw
Halin graphs without any shift along the skirting cycle. We establish hamil
amalgams constructed from cubic Halin graphs. We give a sufficient condition
tonicity of non-cubic amalgams and characterize infinite classes of non-Hamilto
gams. We also characterize hamiltonicity of amalgams constructed by shifting t
nent Halin graphs by one and of general amalgams of higher degree.

1. Definitions and preliminary results

We consider simple graphs without self-loops. The number of neighbor v
a given vertex is called its *degree*. A *tree* is a connected acyclic graph, wit
of degree greater than 1 called *internal vertices* and degree 1 vertices call
An internal vertex with at most one internal neighbor is called a *remote*
tree with all internal vertices of the same degree d is called d-regular. A
tree is called *cubic*. In a rooted tree, a *subtree* rooted in an internal
includes v and all its descendants. The *down-degree* d of such a verte
number of neighbors in the direction away from the root. Subtrees roote
neighbors are called *principal subtrees* of v. For principal subtrees $S_1, \ldots$
we will denote the subtree rooted in v by $(S_1, \ldots, S_d)$.

A *plane* graph is a graph that has been embedded onto the plane
crossing edges. Given a plane tree T with no degree 2 vertices, the *Halin*
$T, \mathcal{H}(T)$, is the graph that results from the union of T and the *skirting c*

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Fig. 1. A Halin graph (solid vertices) and its outerplanar weak dual (a), a Hamilt
and the corresponding facial cover (b)

Fig. 2. An amalgam and its Hamiltonian cycle resulting from the merge of two H
cycles in the component Halin graphs

its leaf vertices that preserves their order on the periphery of T, see Figu
face of a plane graph is a connected plane region bounded by edges of t
Two faces are *independent* if their boundaries do not share a vertex.
graphs are Hamiltonian, *i.e.*, have a simple cycle that includes all vertic
Hamiltonian cycle in a Halin graph corresponds uniquely to a cover of a
vertices of the graph by a collection of independent faces [6], see Figure
call such a collection a *facial cover* of the Halin graph.

The *weak dual* of a plane graph G is the graph G^* with (finite) faces o
vertices, two vertices being adjacent if the corresponding faces of G
boundary edge. An *outerplanar graph* has a planar embedding in which a
lie on the outer (infinite) face. A graph is *maximal outerplanar (mop)* if
tion of any edge would violate its outerplanarity.

Lemma 1. [1, 3] *The weak dual graph of a Halin graph is outerplanar.*

Given two plane trees with the same number of leaves and no degree 2
T_1 and T_2, and a bijection φ between their leaf sets which preserves thei
the plane, the *amalgam* $A = \mathcal{A}(T_1, T_2, \varphi)$ is the union of the correspond
graphs $\mathcal{H}(T_1)$ and $\mathcal{H}(T_2)$ in which the leaf vertices v and $\varphi(v)$ are identi
be the identity mapping. Figure 2 shows the amalgam $A = \mathcal{A}(T, T, \iota)$ fo
of the Halin graph in Figure 1(a).

Since all Halin graphs are Hamiltonian, it is reasonable to ask a
hamiltonicity of amalgams. In this paper, we focus on hamiltonicity of a
of the form $A = \mathcal{A}(T, T, \gamma_k)$, where γ_k is a cyclic k-shift of the leaves of
$\iota = \gamma_0$). One approach to the construction of a Hamiltonian cycle of an
is by merging Hamiltonian cycles of the two component Halin graphs (t
covers). Given Hamiltonian cycles C_1 and C_2 in the component Halin
the amalgam A, we define the *merge* of the two cycles as the graph $\mathcal{M}(C_1$
the same vertex set as A and the edge set being the union of edges of C
without the edges of the skirting cycle that belong to only one of the
or C_2. In general, $\mathcal{M}(C_1, C_2)$ is not necessarily a cycle. The following
answers the hamiltonicity question for $k = 1$ for a general tree T.

Theorem 2. [4] *For any plane tree T, the amalgam $A = \mathcal{A}(T, T, \gamma_1)$ has
tonian cycle that is the merge of copies of the same Hamiltonian cycle
component Halin graphs isomorphic to $\mathcal{H}(T)$.*

Proof. No two consecutive edges of the skirting cycle are on the boun
faces of any given facial cover of a Halin graph. Thus, it is possible
instances of the same Hamiltonian cycle (corresponding to a facial cov
two copies of $\mathcal{H}(T)$ into a Hamiltonian cycle of A.

We extend this result in the next section by investigating hamilt
amalgams constructed from cubic trees. In the subsequent section, we
amalgams constructed from general trees. There, we expand on stateme
regarding hamiltonicity of general amalgams of high vertex degree.

Lemma 3. [4] *Any amalgam with minimum vertex degree at least 4 is Ha*

For amalgams that are Eulerian, Hamiltonian cycles can be constru
such cycles of the component Halin graphs.

Lemma 4. [4] *Any Eulerian amalgam has a Hamiltonian cycle that is the
two Hamiltonian cycles corresponding to facial covers of the compon
graphs.*

2. Cubic trees

Our interest in amalgams constructed from cubic trees stems from the
any amalgam with minimum degree at least 4 is Hamiltonian [4]. We
acterize hamiltonicity of amalgams constructed from Halin graphs of
cubic trees with small (by 0 or 1) circular shifts of vertices. Given a cu
and its isomorphic copy, we will denote by $\hat{w}$ the vertex in the copy t
sponds to a vertex w in T.

Theorem 5. *Given a cubic tree T, the amalgam $A = \mathcal{A}(T, T, \gamma_0)$ has a Ha
cycle that results from merging two Hamiltonian cycles, one from eac
$\mathcal{H}(T)$.*

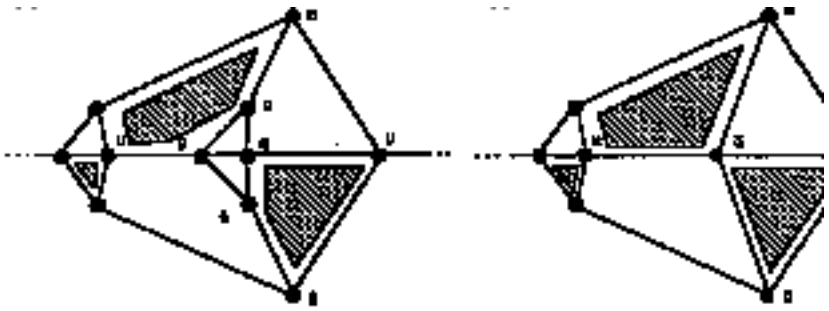

Fig. 3. The amalgams A (a) and A' (b) with the corresponding facial cov

Proof. The theorem follows by induction. The smallest such
$\mathscr{A}(K_{1,3}, K_{1,3}, \gamma_0)$ has a Hamiltonian cycle with no two consecutive edg
skirting cycle corresponding to different facial covers of the compon
graphs. Every cubic tree T of more than four vertices can be reduced to
series of operations that collapse a remote vertex and its two adjacent l
single leaf vertex. For a remote vertex a of T, let T' denote the cubic tre
from the collapse of a and its adjacent leaves p and q into the vertex $\bar{a}$
show that if $A' = \mathscr{A}(T', T', \gamma_0)$ has a Hamiltonian cycle corresponding
covers of the component Halin graphs with no two consecutive edg
skirting cycle, so does $A = \mathscr{A}(T, T, \gamma_0)$ (see Figure 3). Let e be the
neighbor of a in T and u and v be the neighbors of vertex $\bar{a}$ on the skirtin
A'. Consider two cases of the Hamiltonian cycle of A' passing through $\bar{a}$

(i) $(e, \bar{a}), (\bar{a}, \hat{e})$. Then in $\mathscr{A}(T, T, \gamma_0)$, the corresponding Hamiltonian
be obtained by replacing the two edges by the path $(e, a, p, q, \hat{a}, \hat{e})$ or
$(e, a, q, p, \hat{a}, \hat{e})$, depending on whether the edge $(u, \bar{a})$ or the edge $(v, \bar{a})$
Hamiltonian cycle of $\mathscr{H}(T')$ (the graph in Figure 3 illustrates the latter

(ii) $(u, \bar{a}), (\bar{a}, e)$. The corresponding Hamiltonian cycle in $\mathscr{A}(T, T, \gamma_0)$
the path $(u, p, \hat{a}, q, a, e)$.

The amalgam $A = \mathscr{A}(T, T, \gamma_0)$ can be viewed as a special case of sh
isomorphic copy of T along the skirting cycle. We will consider the ham
question for $\mathscr{A}(T, T, \gamma_k)$, where T is cubic and γ_k is the circular shift b
orems 2 and 5 have resolved the issue for $k = 0$ and 1.

We can view Hamiltonian cycles of cubic Halin graphs (equivalen
covers of its vertices) in terms of vertex colorings of its weak dual, whic
mal outerplanar.

Lemma 6. *There is a bijection between the set of cubic trees, the correspond
graphs, and the set of maximal outerplanar graphs.*

Proof. Each mop has the associated cubic tree as its weak dual (see [1,
versely, the weak dual graph H^* of a Halin graph $H = \mathscr{H}(T)$ is a max
erplanar graph; each face of H shares an edge with the outer face, an
ternal face of H^* is triangular, since T is cubic.

Theorem 7. *Every cubic Halin graph has exactly three independent facial*

Proof. The weak dual H^* of a cubic Halin graph H is a maximal ou
graph. As such, H^* has a unique 3-coloring of vertices ([2], Corollary
Each monochromatic vertex set corresponds to a facial cover of H.

Given a cubic Halin graph H of an n-leaf tree, consider labeling the
the skirting cycle by the color (in the above sense) of the adjacent fa
defines an n-character circular string $\sigma \in \{1,2,3\}^n$, such that each of the
phabet symbols is present, and no two adjacent string characters are the s
string σ represents the labeling of cycle edges of H. Two strings are equ
one can be obtained from the other by a permutation of symbols (*i.e.*, by
the corresponding facial covers).

For *complete* cubic trees, in which all leaves are at the same distance
central vertex, we find an easy resolution of the hamiltonicity questio
associated amalgam.

Theorem 8. *For a complete cubic tree T and any integer k, the amalgam $\mathscr{A}$ has a Hamiltonian cycle that is the result of merging two Hamiltonian cyc two copies of $\mathscr{H}(T)$.*

Proof. It is easy to see by induction on the height of T that the string σ r
ing independent facial covers of $\mathscr{H}(T)$ is of the form $\langle (123)^{n/3} \rangle$. Given
periodicity of σ, we need only consider two cases, $k = 0$ and $k = 1$. Th
have been resolved in Theorems 2 and 5 for a general cubic tree T. For a
cubic tree, any circular shift of σ allows choosing symbols i and j, i, j
so that they do not appear as the same character of σ and $\gamma_k(\sigma)$, respect
$k = 0$, any two Hamiltonian cycles corresponding to independent facia
and j, $i \neq j$ will work, while for $k = 1$, we need $i \neq j - 1 \bmod 3$.

We observe a correspondence between restricted ternary strings (rep
facial covers of a graph) and cubic Halin graphs.

Lemma 9. *For every ternary string σ of length n with all three symbols pr no two adjacent characters identical, there is a Halin graph H of an n-leaf such that σ corresponds to the labeling of cycle edges of H by the three f erings of H.*

Proof. We will construct an n-vertex maximal outerplanar graph M w
coloring of vertices defined by σ. We start with an n-cycle C colored acc
σ. If $n = 3$, the only M is the triangle, K_3. For $n \geq 4$, there are at least tw
u and v of C of the same color $i \in \{1,2,3\}$. Take two non-adjacent verti
remaining colors, separated by u and v, and subdivide C by a chord c
these vertices into two cycles, each of fewer than n vertices. The substrin
σ_2 of σ defined by these two cycles have the required property and th
ductive assumption, correspond to mops M_1 and M_2 with the corresp
colorings. The union of M_1 and M_2 identifying the end vertices of th
chord results in a mop that has a 3-coloring defined by σ. By Lemma 6,
unique Halin graph corresponding to M with facial covers represented b

One might expect that for every cubic tree T (not necessarily comp
integer k, there exists a Hamiltonian cycle in the amalgam $A = \mathscr{A}(T, T,$
a merge of Hamiltonian cycles in the component Halin graphs. Unfortun
possible to construct an n-string σ so that there exists k, $1 < k < n-1$,
in σ and $\gamma_k(\sigma)$, for each pair $i, j \in \{1,2,3\}^2$ there is a position $l, 1 \leq l$
that $(\gamma_k(\sigma)[l], \sigma[l]) = (i, j)$. This means that for any choice of a Hamilto
in each of the two component Halin graphs, there is an edge of the
skirting cycle in A that belongs to none of them. Hence, there is no Ha
cycle of A that is the result of merging two Hamiltonian cycles in the co
Halin graphs. A small example with $n = 18$ and $k = 2$ is given below
stands for any suitable symbol, different from its neighbors):

$$\sigma = 1x1x2x2x3x3x2x1x3x$$

This example also contradicts the expectation that hamiltonicity of am
similar to Halin graphs, *i.e.,* in that every Hamiltonian cycle corresp
facial cover of vertices. We have established, by inspection, that the
corresponding to one realization of the above string σ is Hamiltonian. T
tence of a Hamiltonian cycle in an amalgam $\mathscr{A}(T, T, \gamma_k)$ does not depe
ability to merge two Hamiltonian cycles in the component Halin graphs.

3. General trees

In this section, we discuss hamiltonicity of amalgams with minimum vertex
We first generalize Theorem 5 for the case of regular trees and then i
substructures that deny hamiltonicity of amalgams.

Theorem 10. *Given a regular tree T with internal vertex degrees at le
amalgam $A = \mathscr{A}(T, T, \gamma_0)$ has a Hamiltonian cycle that is the result of me
facial covers of the component Halin graphs.*

Proof. The proof is similar to that in Theorem 5. A tree T is either a *sta*
some $k > 3$) or can be reduced to a star by a series of operations that
remote vertex and its adjacent leaves to a single leaf vertex. For a remot
of T, let T' denote the tree resulting from the collapse of a and its adjac
into the vertex $\bar{a}$. We will show that if A' has a Hamiltonian cycle corre
to facial covers of the component Halin graphs, so does A. (Note that
require that no two consecutive edges of the Hamiltonian cycle are on th
cycle C, as is necessary in the proof of Theorem 5.) We maintain the
introduced in Figure 3 with some additional leaf vertices between p an
Let e be the non-leaf neighbor of a in T, and let u and v be the neighbor
vertex $\bar{a}$ on the skirting cycle of A'. Consider three cases of the Hamilto
of A' passing through $\bar{a}$:

(i) $(e, \bar{a}), (\bar{a}, \hat{e})$. The corresponding Hamiltonian cycle in A can be
by replacing the two edges with the path $(e, a, p, \ldots, q, \hat{a}, \hat{e})$ or
$(e, a, q, \ldots, p, \hat{a}, \hat{e})$, depending on whether the edge $(u, \bar{a})$ or the edge $(v, \bar{a}$
Hamiltonian cycle of $\mathscr{H}(T')$;

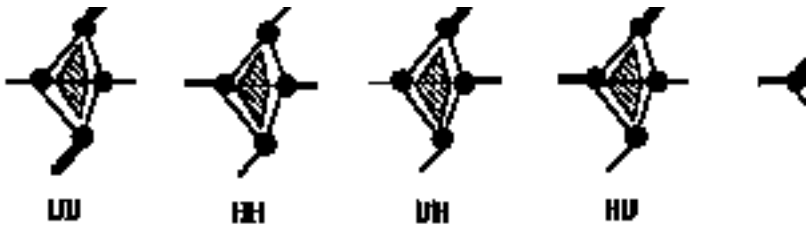

Fig. 4. The five cases of a Hamiltonian cycle traversing a diamond in an am

(ii) $(u,\bar{a}),(\bar{a},e)$. The corresponding Hamiltonian cycle in A will ha path of the form $(u,p,\ldots,r_i,\hat{a},r_{i+1},\ldots,q,a,e)$, where r_i and r_{i+1} are consecutive leaves between p and q (inclusive);

(iii) $(u,\bar{a}),(\bar{a},v)$ (the additional case, not present in the proof of Th The corresponding Hamiltonian cycle in A will have a subpath of $(u,p,\ldots,r_i,a,r_{i+1},\ldots,r_j,\hat{a},r_{j+1},\ldots,q,v)$, where r_i and r_j are any two consecutive) leaves between p and q (inclusive).

The theorem follows by induction from the fact that the smallest su gam $\mathcal{A}(K_{1,k},K_{1,k},\gamma_0)$ has a Hamiltonian cycle corresponding to facial the component Halin graphs.

We now describe construction of a Hamiltonian cycle of the amal $\mathcal{A}(T,T,\gamma_0)$ for an arbitrary tree T. We do it by arbitrarily rooting T versing all vertices of the corresponding subtrees in the two copies of T (s about the skirting cycle). We refer to this rooted tree as $\bar{T}$. We observe t ces of any (proper, non-trivial) subtree S of $\bar{T}$ and its isomorphic copy rated from the remainder of A by four vertices (called, w.l.o.g., *upper*, and *lower* vertices). These vertices determine the *diamond* $d(S)$. There are ways that a Hamiltonian cycle of A can traverse a diamond (below, H a nify horizontal and vertical access to the separator vertices, left or right a or lower, respectively), as follows:

VV: from the upper to the lower vertex,
HH: from the left to the right vertex,
VH: from the upper (or lower) vertex to the right vertex,
HV: from the left to the upper (or lower) vertex,
X: in two segments: from the left to upper and from the right to lowe (or *vice versa*).

We observe that there is no directionality of the Hamiltonian cycle; "from" and "to" above can be reversed. Because of the symmetry a skirting cycle, the specification of upper and lower vertices is arbitrary.

We can now define the five traversals of diamonds for a proper sul $(S_1 \ldots S_d)$ of $\bar{T}$. The strings below use as components *local* traversals of subdiamonds, with "(HH)*" denoting zero or more repetitions of l traversal.

Lemma 11. *The five traversals of the diamond for a general subtree $S = (s$ $d > 1$, can only be realized by the following sequences of traversals of the subdiamonds $d(S_i)$:*

HH: ⟨HH⟩* ⟨HV⟩⟨VV⟩⟨VH⟩⟨HH⟩* or ⟨HH⟩*⟨X⟩⟨HH⟩*⟨X⟩⟨HH⟩* or ⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩*⟨X⟩⟨HH⟩* or ⟨HH⟩*⟨X⟩⟨HH⟩*⟨HV⟩⟨VH⟩ or ⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩*,
VH: ⟨VV⟩⟨VH⟩⟨HH⟩* or ⟨VH⟩⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩* or ⟨VH⟩⟨HH⟩*⟨X⟩⟨HH⟩*,
HV: ⟨HH⟩*⟨HV⟩⟨VV⟩ or ⟨HH⟩*⟨X⟩⟨HH⟩*⟨HV⟩ or ⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩*⟨HV⟩,
X: ⟨HH⟩*⟨X⟩⟨HH⟩* or ⟨HH⟩*⟨HV⟩⟨VH⟩⟨HH⟩*,
VV: ⟨VH⟩ ⟨HH⟩* ⟨HV⟩.

Proof. The traversals requiring entry ⟨exit⟩ through an upper or lower v the most restrictive, as they can provide connection to only one princ diamond. In the case of VV, the path from the upper vertex has to pas the first or last principal subdiamond, and symmetrically for the lower ve same restriction of connecting only one principal subdiamond each with and the lower vertices applies in the case of X traversal, where "vertical s the path can be performed by the ⟨X⟩ or by the ⟨HV⟩⟨VH⟩ traversals o principal subdiamonds, resulting in the two possible sequences.

In the case of HV (a symmetrical argument applies to VH), one of and upper vertices allows two principal subdiamonds to be connected b versal. The left of those two can be traversed by an ⟨X⟩ traversal, in whic rightmost principal subdiamond is necessarily traversed by ⟨HV⟩, or by traversal, which requires in addition that the adjacent principal subdia traversed by a ⟨VH⟩ traversal, unless it is also the rightmost principal sub in which case it has to be traversed by a ⟨VV⟩ traversal.

The case of HH allows the most varying subdiamond traversals. Al share the possibility of ⟨HH⟩ traversals of some number of principal subd Traversal of the upper and lower vertices of the diamond can be accom "vertical swings" of the path from the skirting cycle through the ⟨HV⟩⟨VV⟩⟨VH⟩ (both vertices traversed), an ⟨X⟩ traversal (accessing b ces), or the sequence ⟨HV⟩⟨VH⟩ (traversing one vertex or connecting ea with a path segment). Combinations of the latter two lead to the fo sequences.

The necessity of these traversals follows by an inductive proof on the of the root of subtree S. Since ⟨HH⟩* traversals can be inserted betwee ible traversals of two adjacent subdiamonds, we need only consider traversals, limiting the degree d in the traversals: to 2 in ⟨X⟩ and ⟨VV ⟨VH⟩ and ⟨HV⟩, and to 4 in ⟨HH⟩. These cases can be resolved by inspe

Figure 5 illustrates some of the traversals listed in the above lemma.

Lemma 12. *For a remote vertex of down-degree $d > 1$ with all d subtrees t rooted tree $\overline{T}$, all the above traversals can be realized with the exception o $d = 2$. For the trivial subtree of $\overline{T}$ ($d = 1$), the traversal X cannot be reali*

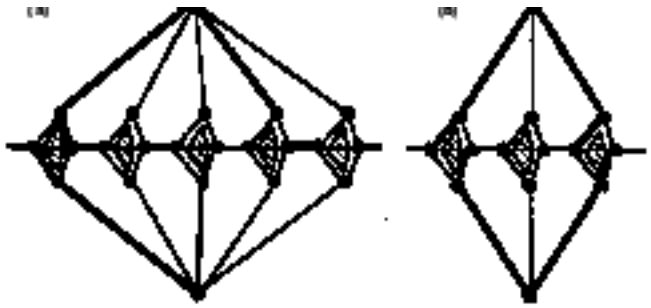

Fig. 5. The sequences (X)(HH)(HV)(VH)(HH) (a) and (X)(HH)(X) (b) realizi traversal of a diamond in an amalgam

Proof. The traversal X is not possible in the "trivial diamond" ($d =$ requires two disjoint paths through the diamond. In the four-vertex ($d = 2$), entering at the left (right) vertex precludes traversal of the upper vertex, unless the traversing path leaves the diamond through one of thes This allows X, VH, HV, and VV, but not HH. In diamonds with the rem having higher degree, all five traversals can be accomplished, as ind Lemma 11.

To simplify our discussion of the existence of traversals defining a Ha cycle in an amalgam, we observe the following:

Lemma 13. *In every Hamiltonian amalgam $A = \mathcal{A}(T, T, \gamma_0)$, a Hamilto must include an edge of the skirting cycle between two adjacent subtrees o*

Proof. Assume that no edge of the skirting cycle between subdiamond traversed by any Hamiltonian cycle of A. We can therefore remove th hamiltonicity of the remaining graph would imply hamiltonicity of K_2 contradicting the well-known fact that such graphs are non-Hamiltoni any Hamiltonian cycle of A must include an edge of the skirting cycle.

By the above lemma, we need only consider the case HH of traversi amond of a tree T, independent of its rooting. We must consider each tween adjacent subtrees as the possible edge of the Hamiltonian cycle.

We can further characterize Hamiltonian amalgams with minimum gree 3 by considering the location of degree 3 vertices.

Theorem 14. *Any amalgam $A = \mathcal{A}(T, T, \gamma_0)$ for a tree T with no vertex o adjacent to a leaf vertex is Hamiltonian.*

Proof. We prove the theorem by showing that any diamond correspon subtree T' of T can be traversed in each of the five ways introduced ab proof will be by induction on the length h of the longest path from the r diamond to a vertex of the skirting cycle (*i.e.*, the *height* of the tree T'). case ($h = 1$, the root is a remote vertex of down-degree at least 3) can be in all five ways by Lemma 11.

We now assume that for $h > 1$, the diamond corresponding to all height less than h with no degree 3 vertex adjacent to a leaf can be traversed of the ways as above. We consider the case of a tree with height h. We two cases of the down-degree d of the root, as follows:

(i) $d = 2$. Each of the two principal subtrees of T' has height at least inductive assumption, each subtree can be traversed in all ways. As su traversable as follows: ⟨VH⟩⟨HV⟩ realizing VV traversal of T, ⟨X⟩⟨X⟩ HH, ⟨VV⟩⟨VH⟩ realizing VH, ⟨HV⟩⟨VV⟩ realizing HV, and ⟨HV⟩⟨VH⟩ X.

(ii) $d \geq 3$. It suffices to consider three principal subtrees of T', any s subtree to be traversed by HH. By the inductive assumption, the only hindrance to a traversal can come from the attempt to traverse the trivi by X. Since one of the above three subtrees must be non-trivial (as $h >$ each of the five traversals we can find a realization that does not involve T' can be traversed in all five ways.

The hamiltonicity of A follows from the above and Lemma 13.

By the above theorem, we restrict amalgams that may not be Hamil those that have degree 3 vertices adjacent to the skirting cycle. We furt acterize non-Hamiltonian amalgams in the following theorem.

Theorem 15. *All non-Hamiltonian amalgams of the form $\mathscr{A}(T, T, \gamma_0)$ can by rooted trees with a degree 3 root, one trivial (one vertex) principal su one principal subtree that does not allow any of the traversals* VV, VH, H

Proof. Consider the tree T that defines a non-Hamiltonian amalg $\mathscr{A}(T, T, \gamma_0)$. Since T must have a degree 3 vertex adjacent to a leaf, we such a vertex of minimum height as the root of $\overline{T}$, leading to two possib urations, depending on whether the vertex is adjacent to one or two lea all three neighbors are leaves, since then we would have $T = K_{1,3}$ for w Hamiltonian.) There are only six possible HH traversals of a diamond w 3 root: ⟨X⟩⟨X⟩⟨HH⟩, ⟨HH⟩⟨X⟩⟨X⟩, ⟨X⟩⟨HH⟩⟨X⟩,⟨HV⟩⟨VH⟩⟨X⟩, ⟨HV⟩⟨' or ⟨X⟩⟨HV⟩⟨VH⟩. To deny hamiltonicity of A in the case of two adjace the nontrivial subtree of $\overline{T}$ must not be traversable by either VH, HV, the two leaves do not allow an X traversal, precluding the first three HH above. In the case of one adjacent leaf, the vertices of one of the subtrees be traversed in any manner; the other non-trivial subtree again must not of the four traversals.

4. Non-Hamiltonian amalgams

Lastly, we consider several small examples of non-Hamiltonian a $\mathscr{A}(T, T, \gamma_0)$. We focus on the structure of tree T, rooted at some arbitra remembering there is an isomorphic copy of T intersecting at a skirting cy associated amalgam, as described above. We will consider trees T height. By definition of an amalgam, the root of tree T must be of de

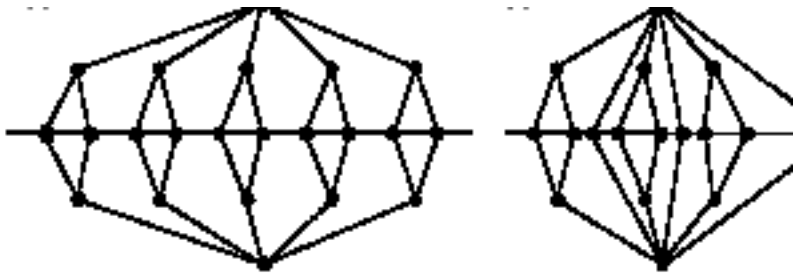

Fig. 6. Two small examples of non-Hamiltonian amalgams $\mathcal{A}(T, T, \gamma_0)$

higher. Recall that the amalgam associated with a tree T is Hamiltoni only if the diamond corresponding to a tree isomorphic to T as the pla tree is HH traversable.

First, we consider trees T of height 1. By our previous discussion, w all relevant amalgams are Hamiltonian, as we can avoid any impossibl traversal of a trivial subtree to realize an HH traversal.

Next, we consider trees T of height 2. We classify the subtrees of t trivial (a single vertex on the skirting cycle, denying a local X traversal), degree 2 (denying a local HH traversal), or of higher down-degree (allo traversal). By definition, a tree T of height 2 has at least one non-trivia For a root of degree at most 4, all associated amalgams are Hamiltonian exist HH traversals not requiring a local X or HH traversal of any subtr we consider a degree 5 vertex, any HH traversal of tree T must involv HH traversal of at least one subtree. Thus, if all subtrees are of down- the associated amalgam is not Hamiltonian. Figure 6(a) presents Hamiltonian amalgam. If any of the subtrees is trivial or has a root down-degree, that subtree can be HH traversed first, reducing the proble of down-degree 4, yielding an amalgam that is Hamiltonian.

Next, consider T of height 2 with root of degree 6 or higher. We see HH traversal will require two or more HH traversals. If the root has at l subtress with roots of down-degree 2 that are separated by at least one subtree regardless of rotation, then the associated amalgam is not Ham There exists no HH traversal of T, as it would require an HH traversal of with down-degree 2, which is impossible. Figure 6(b) shows the smal Hamiltonian amalgam $\mathcal{A}(T, T, \gamma_0)$ based on one of these trees T.

Finally, we consider trees of greater height. One set of non-Hamilton lagams is generated according to the last pattern noted above. As lon amalgams have three separated subtrees with roots of down degree 2 ac the skirting cycle, they are not Hamilitonian. We find another tree T o having a root of degree 3, as shown in Figure 7. Each subtree of T prec versal by X or VV. Thus, there are no HH traversals of T, and the a amalgam is not Hamiltonian. One can add an arbitrary principal subtr structure and maintain the non-hamiltonicity.

Fig. 7. A tree T of height 3 resulting in a non-Hamiltonian amalgam $\mathcal{A}(T,$

5. Conclusions

In this paper, we have characterized several classes of Hamiltonian Hamiltonian amalgams. We establish hamiltonicity of all amalgams in vertices are of degree 4 or more. All amalagams of the form $\mathcal{A}(T,$ Hamiltonian. For the case of $\mathcal{A}(T, T, \gamma_0)$, we establish hamiltonicity of based on cubic trees T. We also characterize amalgams $\mathcal{A}(T, T, \gamma_0)$ tha Hamiltonian and provide several small examples. A challenging researc mains to find a complete set of smallest examples of non-Hamiltonian that are substructures of all such amalgams.

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