# Embeddings of $k$-connected graphs of pathwidth $k$ *

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**Abstract**

We present $O(n^3)$ embedding algorithms (subgraph isomorphism and its generalizations) for classes of graphs of bounded pathwidth, where $n$ is the number of vertices in the graph. These include the first polynomial-time algorithm for minor containment and the first $O(n^c)$ algorithm ($c$ a constant independent of $k$) for topological embedding of graphs from subclasses of partial $k$-trees, as well as an $O(n^2)$ algorithm for subgraph isomorphism. Of independent interest are structural properties of $k$-connected graphs of bounded pathwidth on which our algorithms are based. We also describe special cases which reduce to various generalizations of string matching, permitting more efficient solutions. Finally, we describe $n^{k+O(1)}$ algorithms for solving these problems on any graphs of pathwidth at most $k$.

## 1 Introduction

Many fundamental problems in a diverse set of research areas can be characterized as graph embedding problems, where data is represented as graphs and patterns can be detected by finding smaller graphs in larger ones. Classic pattern-matching problems make use of the subgraph isomorphism problem, namely, the problem of determining whether there is a subgraph of an input graph $H$ that is isomorphic to an input graph $G$. Viewed as an injective mapping, the subgraph isomorphism of $G$ into $H$ consists of a mapping of vertices of $G$ to vertices of $H$ so that edges of $G$ map to corresponding edges of $H$. Generalizations of this mapping include topological embedding, where vertices of $G$ map to vertices of $H$ and edges of $G$ map to vertex-disjoint paths in $H$, and minor containment, where vertices of $G$ map to disjoint connected subgraphs of $H$ and edges of $G$ map to edges of $H$.

Subgraph isomorphism (and therefore its generalizations listed above) is known to be $\mathcal{NP}$-complete for general graphs, but can be solved in polynomial time for many restricted classes of graphs. Of particular interest are partial $k$-trees, also known as graphs of bounded treewidth (to be defined formally in Section 2), algorithms for which unify many of the known polynomial-time algorithms for embedding problems. The embedding problems are also $\mathcal{NP}$-complete for general partial $k$-trees (implied by [Sys82]), even under many different connectivity and degree bounds on both graphs [GN96a].

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| Problems/classes | $G$, $H$ proper | any $G$ , $H$ full | $G$ full, any $H$ | any $G$ and $H$ |
|---|---|---|---|---|
| subgraph isomorphism | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ |
| topological embedding | $O(n^3)$ | $O(n^3)$ | $O(n^3)$ | $n^{k+O(1)}$ |
| minor containment | $O(n^3)$ | $O(n^3)$ | $n^{k+O(1)}$ | $n^{k+O(1)}$ |

Figure 1: Upper bounds on problems for classes of $k$-connected partial $k$-paths

However, when both $G$ and $H$ are $k$-connected partial $k$-trees, there are polynomial-time algorithms for both subgraph isomorphism [MT92, DLP00] and topological embedding [GN94, GN98] but minor containment remains $\mathcal{NP}$-complete even for this restricted class [MT92].

The state of our knowledge about these problems is unsatisfying in a number of ways. The degree of the polynomial in the complexity of the algorithms for subgraph isomorphism and topological embedding depends on the magnitude of $k$. This raises the question of whether there is an algorithm that runs in time $O(n^c)$ for $c$ a constant independent of $k$. (Such an algorithm would be unlikely if the problems were fixed-parameter intractable [DF95]). Furthermore, although polynomial-time algorithms for minor containment have been obtained when there is a degree bound [MT92, GN95], there are no previous results relating connectivity constraints and polynomial-time minor containment algorithms.

Our contributions in this paper demonstrate that for large subclasses of $k$-connected graphs of bounded pathwidth (a restriction on partial $k$-trees), there exist $O(n^3)$ algorithms for minor containment and topological embedding, and $O(n^2)$ algorithms for subgraph isomorphism (these are summarized in Figure 1). The results are of particular interest in light of the fact that all the problems are $\mathcal{NP}$-complete when both graphs have connectivity less than $k$ [GN96a]. The algorithms make use of a new and elegant characterization of $k$-connected graphs of bounded pathwidth (Section 3) which allows us to form a common framework for the algorithms (Sections 4, 5, and 6). We show that each such graph has an essentially unique layout of the vertices on $k$ "tracks". These layouts, and the restrictions they imply on the structure of topological embeddings and minor containments, allow the description of intuitive algorithms with elegant proofs of correctness. Removing the restriction to the subclasses, we describe (Section 7) $n^{k+O(1)}$ algorithms for the problems on graphs of pathwidth bounded by $k$. Finally, in special cases (Section 8) we can exploit further structure to reduce the problems to string matching and its variants, permitting more efficient solutions.

# 2 Preliminaries

## 2.1 Graphs, treewidth, and pathwidth

Throughout this paper we use standard graph-theoretic notation [BM76]. The vertex and edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively; we use $n$ to denote $|V(G)|$. All graphs we consider are simple, connected, and without self-loops. The set of vertices adjacent to a vertex $v$, the neighbourhood of $v$, is denoted by $N(v)$. A graph is *$k$-connected* if there are $k$ vertex-disjoint paths between every pair of its vertices. Menger's Theorem states that any separator of a $k$-connected graph (a set of vertices whose removal disconnects the graph) contains at least $k$ vertices.

In this paper we deal with subclasses of graphs of bounded treewidth, as defined below.

**Definition 1** *A* tree decomposition *of a graph $G$ is a pair $(T, \chi)$ where $T$ is a tree and $\chi : V(T) \to 2^{V(G)}$ satisfies the following three properties:*

1. *for every $a \in V(G)$, there is an $x \in V(T)$ such that $a \in \chi(x)$;*

2. *for every $e = (a, b) \in E(G)$, there is an $x \in V(T)$ such that $a, b \in \chi(x)$; and*

3. *for all $x, y, z \in V(T)$, if $y$ is on the path from $x$ to $z$ in $T$ then $\chi(x) \cap \chi(z) \subseteq \chi(y)$.*

The *width* of a tree decomposition $(T, \chi)$ is $\max\{|\chi(x)| - 1 : x \in V(T)\}$. The *treewidth* of a graph $G$ is the minimum width over all its tree decompositions; a *graph of bounded treewidth* is a graph of treewidth $k$ for some constant $k$. For $p$ a vertex of $T$, $\chi(p)$ is called a *bag* of $T$. A *path decomposition* of a graph is a tree decomposition in which $T$ is a path, and the *pathwidth* of $G$ is the minimum width over all its path decompositions. For fixed $k$, decompositions of treewidth or pathwidth $k$ can be found in linear time [Bod93]. By property 3 above, every bag $\chi(p)$ (other than for $p$ a leaf) is a separator of $G$ as the removal of its vertices disconnects $G$.

There is an equivalent definition of graphs of bounded treewidth which is often useful. A *$k$-tree* (sometimes *full $k$-tree*) is either a $(k + 1)$-clique, or a graph formed from a smaller $k$-tree by adding a new vertex $v$ of degree $k$ adjacent to all vertices of a $k$-clique $C$ ($C$ is called the *attachment clique* of $v$, or $\mathrm{AC}(v)$). A *$k$-leaf* is any degree-$k$ vertex of a $k$-tree, and a *partial $k$-tree* is any subgraph of a $k$-tree. Partial $k$-trees are exactly graphs of treewidth at most $k$ [vL90].

A *full $k$-path* is a special type of $k$-tree. In its construction, we maintain the notion of a "current clique" (initially $k$ vertices of the first $(k + 1)$-clique, the remaining vertex being the *initial $k$-leaf*). When a new vertex is added (with the current clique as its attachment clique), it enters the current clique, and one vertex (possibly the new one) leaves the current clique, never to return. Note that if the new vertex immediately leaves, it is a $k$-leaf (the last vertex added being the *final $k$-leaf*). In a *proper $k$-path* [TUK95], the new vertex is not permitted to immediately leave (as a consequence, a proper $k$-path of size at least $k + 2$ has only two $k$-leaves).

Figure 1(a) below illustrates a full 2 path, where $a$ is the initial 2-leaf and $\{b, c\}$ in the initial current clique. The vertices are added in order $d, e, f, g, h, i$, with $i$ being the final $k$-leaf. The graph is not a proper $k$-path since after $f$ is added to attachment clique $\{d, e\}$, it leaves immediately, allowing $g$ to have attachment clique $\{d, e\}$ as well.

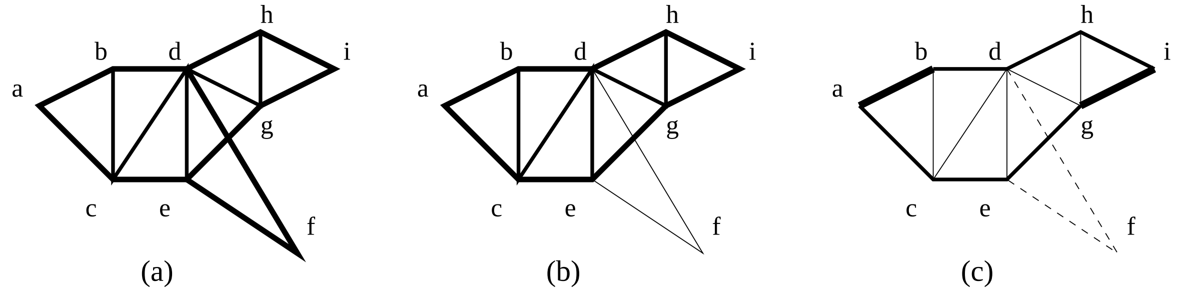


Figure 2: Example of a full 2-path

The class of *partial $k$-paths* (subgraphs of $k$-paths) is equivalent to the class of graphs of pathwidth at most $k$ [Pro89]. The terminology we use here is in common use, though the original use of "$k$-path" in the previous citation was to refer to what we call a proper $k$-path, and other authors [KS96] have used "proper pathwidth" as a synonym for bandwidth. A *partial proper $k$-path* is a subgraph of a proper $k$-path.

We observe that a full $k$-path can be partitioned into the *body*, which is a proper $k$-path that includes the initial and final $k$-leaves, and *hairs*, which are the remaining $k$-leaves and their adjacent edges. We define an *end* of a full $k$-path to be the neighborhood $h$ of a degree-$k$ vertex such that the subgraph of $G$ induced by $V(G) \setminus \{h\}$ has at most one component of size greater than 1. There are at most two possible ends in a full $k$-path of size at least $k + 3$, and the initial and final $k$-leaves each have a distinct end as their neighbor set (the *head* and *tail*, respectively). In Figure 1(b), the body is marked with thick lines and the hair with thin lines; $N(a)$ and $N(i)$ are the ends. We can view a partial $k$-path being decomposed in a similar fashion, where the body is a partial proper $k$-path; in a $k$-connected graph the ends will still be neighborhoods of degree-$k$ vertices.

Our algorithms will make use of a special type of path decomposition, as defined below:

**Definition 2** *A path decomposition* $(P, \chi)$*,* $P = p_1, \ldots, p_\ell$*, of a graph* $G$ *is a* normalized path decomposition *if*

1. $|\chi(p_i)| = k + 1$ *for* $i$ *odd;*
2. $|\chi(p_i)| = k$ *for* $i$ *even; and*
3. $\chi(p_{i-1}) \cap \chi(p_{i+1}) = \chi(p_i)$ *for even* $i$*.*

Notice that $\ell$ is always odd in this definition. It is not difficult to see that such a decomposition can be generated during the construction of a $k$-path; the bags of size $k + 1$ are the attachment cliques plus respective new vertices, and the bags of size $k$ are the current cliques. Given an already-constructed $k$-path, one possible construction sequence can be established by a simple linear-time scan, starting from one end.

Throughout this paper, we assume that $G$ and $H$ are $k$-connected graphs of pathwidth $k$, and that all path decompositions are normalized. We will make use of running examples of 4-connected 4-paths $G$ and $H$ in Figures 2.1 and 2.1. The 4-bags in a normalized path decomposition of $H$ are illustrated in Table 2.1; the first 5-bag is $(u_I, u_1, v_1, w_1, x_1)$, the last is $(u_6, v_4, w_6, x_5, u_F)$, and between each adjacent pair of 4-bags is a 5-bag containing the union of the 4-bags. For ease of exposition, we use letters near the beginning of the alphabet $(a, b, \ldots)$ to denote vertices in $G$, letters near the end of the alphabet $(u, v, \ldots)$ to denote vertices in $H$, and letters in the middle of the alphabet $(p, q, \ldots)$ to denote vertices in a path decomposition. `Hole 19: minimum number of nodes needed in a graph to make lemma work, e.g. Lemma 19.`

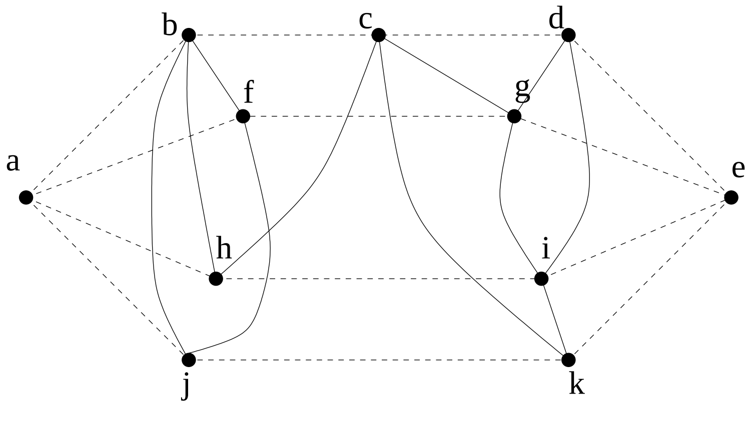


Figure 3: A 4-connected proper 4-path $G$

## 2.2 Embeddings

Each of the embeddings considered in this paper can be defined in terms of injective mappings. A *subgraph isomorphism* maps vertices of $G$ to vertices of $H$ and edges of $G$ to edges of $H$; it is a special case of a *topological embedding*, which maps vertices of $G$ to vertices of $H$ and edges of $G$ to vertex-disjoint paths in $H$ (for subgraph isomorphism, $\phi$ maps edges of $G$ to edges of $H$).

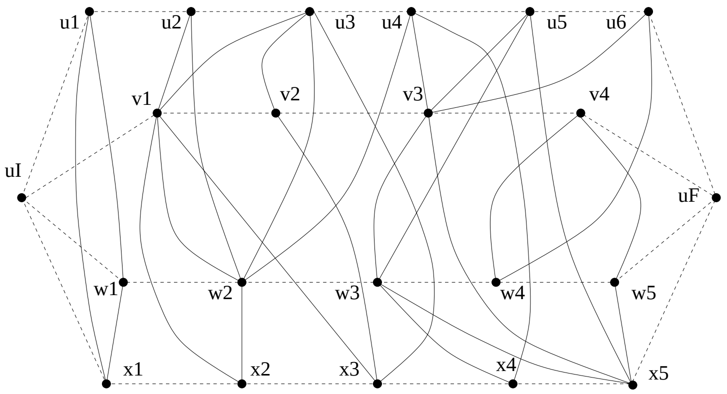


Figure 4: A 4-connected proper 4-path $H$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | $u_1$ | $u_2$ | $u_2$ | $u_3$ | $u_3$ | $u_3$ | $u_4$ | $u_4$ | $u_4$ | $u_4$ | $u_4$ | $u_5$ | $u_5$ | $u_6$ | $u_6$ | $u_6$ |
| $v_1$ | $v_1$ | $v_1$ | $v_1$ | $v_1$ | $v_1$ | $v_2$ | $v_2$ | $v_2$ | $v_2$ | $v_3$ | $v_3$ | $v_3$ | $v_3$ | $v_3$ | $v_4$ | $v_4$ |
| $w_1$ | $w_2$ | $w_2$ | $w_2$ | $w_2$ | $w_2$ | $w_2$ | $w_2$ | $w_3$ | $w_3$ | $w_3$ | $w_3$ | $w_3$ | $w_4$ | $w_4$ | $w_4$ | $w_5$ |
| $x_1$ | $x_1$ | $x_1$ | $x_2$ | $x_2$ | $x_3$ | $x_3$ | $x_3$ | $x_3$ | $x_4$ | $x_4$ | $x_5$ | $x_5$ | $x_5$ | $x_5$ | $x_5$ | $x_5$ |

Table 1: The 4-bags in a normalized path decomposition of $H$

**Definition 3** *The graph $G$ is* topologically embeddable *in the graph $H$ if there is a pair of injective functions $f : V(G) \to V(H)$ and $\phi : E(G) \to \{\text{paths in } H\}$ such that:*

1. *if $e = (a, b) \in E(G)$ then $\phi(e)$ has endpoints $f(a)$ and $f(b)$; and*

2. *for $e, e' \in E(G)$, $e \neq e'$, the only vertices that $\phi(e)$ and $\phi(e')$ can have in common are their endpoints.*

*We call $(f, \phi)$ a* topological embedding of $G$ into $H$*; similarly, for $(f, \phi)$ such that $\phi : E(G) \to E(H)$, we call $(f, \phi)$ a* subgraph isomorphism embedding of $G$ into $H$ .

We can view $(f, \phi)$ as instructions for transforming $H$ into $G$. In this process, we first delete some edges and vertices (*deleted vertices*) to form a subgraph $H'$ of $H$. Next we "shrink" paths of $H'$ into edges, where the endpoints of the resultant edge are the termini of the path. The conversion of a path into an edge can be viewed as collapsing the intermediate vertices on the path (*collapsed vertices*) into one of the resultant edge's endpoints (*preserved vertices*). Figures 6 and 4 illustrate subgraph isomorphism and topological embeddings of $G$ in $H$, respectively.

In discussing properties of embeddings of $G$ into $H$, we will often rely on the fact that if $G$ is embeddable in $H$, we can derive an *induced path decomposition* of $G$ from the path decomposition of $H$; details of this process for topological embedding were developed for $k$-connected partial $k$-trees [GN98]. To facilitate the definition, for a topological embedding $(f, \phi)$ we define a surjective function $\psi$ that inverts $f$ on its image, and maps each interior vertex in $\phi(e)$ to one of the endpoints of $e$. That is, $\psi(f(a)) = a$ for every vertex $a$ of $G$, and for every edge $e = (a, b)$ of $G$, there is a vertex $u$ on $\phi(e)$ such that for $u \neq f(b)$, $Q$ is the subpath of $\phi(e)$ from $f(a)$ to $u$, $R = \phi(e) \setminus Q$, for all vertices $v$ of $Q$, $\psi(v) = a$, and for all vertices $w$ of $R$, $\psi(w) = b$.

**Definition 4** *For $(f, \phi)$ a topological embedding of $G$ into $H$, $\psi$ the associated mapping of vertices in $V(H)$ to vertices in $V(G)$, and $(P, \chi)$ a path decomposition of $H$, we define $\mu : V(P) \to 2^{V(G)}$ as follows:*

*$\mu(p) = \{\psi(u) | u \in \chi(p)$ and either $f(a) = u$ for some $a$ or $u$ appears on $\phi(e)$ for some $e \in E(G)\}$. We form a path $P_G$ from $P$ by removing each vertex $p \in V(P)$ such that $|\mu(p)| = 0$ and by replacing each subpath $q_1, \ldots, q_m$ such that $\mu(q_i) = \mu(q_j)$ for all $i, j$ by a single vertex $q$ with $\mu(q) = \mu(q_1)$. We form $\chi_G$ by restricting $\mu$ to $P_G$. $(P_G, \chi_G)$ is the* path decomposition of $G$ induced by $(f, \phi)$.

A graph $G$ is a *minor* of a graph $H$ if a graph isomorphic to $G$ can be formed from $H$ by a series of edge and vertex deletions and edge contractions. Equivalently, each vertex of $G$ is mapped to a distinct connected subgraph of $H$ and each edge of $G$ to a distinct edge of $H$, as defined below.

**Definition 5** *The graph $G$ is a* minor *of the graph $H$ if there is a pair of functions $(f, \xi)$ such that:*

1. *$f : V(G) \rightarrow \{$connected subgraphs of $H\}$;*
2. *$\xi : E(G) \rightarrow E(H)$ is injective;*
3. *if $e = (a, b) \in E(G)$ then there are vertices $u \in V(f(a))$ and $v \in V(f(b))$ such that $\xi(e) = (u, v)$; and*
4. *for every $a, b \in V(G)$, $a \neq b$, $V(f(a)) \cap V(f(b)) = \emptyset$.*

*We call $(f, \xi)$ a* minor embedding *of $G$ into $H$.*

**Definition 6** *For $(f, \xi)$ a minor embedding of $G$ into $H$ and $(P, \chi)$ a path decomposition of $H$, we define $\mu : V(P) \rightarrow 2^{V(G)}$ as follows: $\mu(p) = \{a \in V(G) | u \in V(f(a))$ for some $u \in \chi(p)\}$. We then restrict $P$ to $P_G$ and $\mu$ to $\chi_G$ as in the above definition. $(P_G, \chi_G)$ is the* path decomposition of $G$ induced by $(f, \xi)$.

A minor embedding of $G$ into $H$ is illustrated in Figure 5; induced path decompositions of $H$ resulting from the embeddings depicted in Figures 6, 4, and 5 and shown in Tables 2.2 through 5.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | $u_1$ | $u_2$ | $u_2$ | $b$ | $b$ | $b$ | $c$ | $c$ | $c$ | $c$ | $c$ | $d$ | $d$ | $u_6$ | $u_6$ | $u_6$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $f$ | $f$ | $f$ | $f$ | $g$ | $g$ | $g$ | $g$ | $g$ | $v_4$ | $v_4$ |
| $w_1$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $i$ | $i$ | $i$ | $i$ | $i$ | $w_4$ | $w_4$ | $w_4$ | $w_5$ |
| $x_1$ | $x_1$ | $x_1$ | $x_2$ | $x_2$ | $j$ | $j$ | $j$ | $j$ | $k$ | $k$ | $e$ | $e$ | $e$ | $e$ | $e$ | $e$ |

Table 2: Mapping bags under the subgraph isomorphism embedding

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $b$ | $b$ | $c$ | $c$ | $c$ | $c$ | $c$ | $d$ |
| $a$ | $f$ | $f$ | $f$ | $f$ | $g$ | $g$ | $g$ |
| $h$ | $h$ | $h$ | $i$ | $i$ | $i$ | $i$ | $i$ |
| $j$ | $j$ | $j$ | $j$ | $k$ | $k$ | $e$ | $e$ |

Table 3: Induced path decomposition of $G$ under the subgraph isomorphism embedding

We can now show that for each embedding $(P_G, \chi_G)$ is a normalized path decomposition of width $k$.

**Lemma 1** *For a subgraph isomorphism, topological, or minor embedding of $G$ into $H$, $G$ and $H$ both $k$-connected partial $k$-paths, the induced path decomposition of $G$, $(P_G, \chi_G)$, is a normalized path decomposition of width $k$.*

`Hole 1:  proof of lemma.`

# 3 Track layouts

The additional requirement of $k$-connectivity imposes strong restrictions on the structure of partial $k$-paths. We show that the body vertices can be partitioned into $k$ *tracks* where the tracks form vertex-disjoint paths from the initial $k$-leaf to the final $k$-leaf. This partitioning is unique up to permutation of the tracks, and is independent of any specific path decomposition. Track layouts of full $k$-paths were considered previously [Pro84, PRS98], but our characterization of partial $k$-path embeddings is new.

Tracks can be extracted by examining a path decomposition of the graph. For $(P, \chi)$ a normalized width-$k$ path decomposition of $G$, $P = p_1, \ldots, p_\ell$, we define the *entry vertex* of $p_i$ ($i$ odd, $i > 1$), entry($p_i$), to be the unique vertex in $\chi(p_i) \setminus \chi(p_{i-1})$. Similarly, the *exit vertex* of $p_i$ ($i$ odd, $i < \ell$), exit($p_i$), is the unique vertex in $\chi(p_i) \setminus \chi(p_{i+1})$. A vertex $x$ of $G$ is a *hair vertex* (a non-initial non-final $k$-leaf) exactly when $x = \text{entry}(p_i) = \text{exit}(p_i)$, and hence when entry($p_i$) or exit($p_i$) is a body vertex of $G$, $\text{entry}(p_i) \neq \text{exit}(p_i)$.

For body vertices, we can use exit and entry information to form paths in $G$. If entry($p_i$) is not a $k$-leaf, it must be a neighbor of exit($p_i$) (otherwise $\chi(p_i) \setminus \{\text{entry}(p_i), \text{exit}(p_i)\}$ is a set of size $k-1$ separating entry($p_i$) and exit($p_i$) in $G$). We say that entry($p_i$) *replaces* exit($p_i$). A *track* is a sequence of body vertices $a_1, a_2, \ldots a_t$ such that $a_{i+1}$ replaces $a_i$ for all $1 \leq i < t$. We use the interval notation $[a_i, a_j]$ to represent a segment $a_i, \ldots, a_j$ of a track. The track on which a vertex $a$ appears is denoted by track($a$). For ease of notation, we say that the $k$-leaves are on track 0. A *track edge* of $G$ is any edge adjacent to the initial or final $k$-leaf, or any edge between vertices on the same track. Note that an internal vertex of a track is adjacent to exactly two other vertices on the same track, namely the vertex it replaces, and its own replacement. A *hair edge* is an edge adjacent to a hair. Any edge that is not a track edge or a hair edge is called a *cross edge*. Figure 1(c) illustrates track edges (thick lines), hair edges (dashed lines) and cross edges (thin lines).

Since in our algorithms hairs often need to be handled in a special way, we make use of a special form of path decomposition which distinguishes hairs from body vertices. It is not difficult to see that a compressed path decomposition can be generated from a normalized path decomposition in linear time.

**Definition 7** *$(P, \chi, hair)$, $P = p_2, \ldots, p_{\ell-1}$, is a* compressed path decomposition *of a graph $G$ if*

1. *$hair(p_i)$ is the set of hairs with attachment clique $AC(\chi(p_i))$ for even $i$; and*
2. *for $P' = p_1, p_2, \ldots, p_{\ell-1}, p_\ell$, $a_I$ an arbitrary vertex in $hair(p_2)$, $a_F$ an arbitrary vertex in $hair(p_{\ell-1})$, $\chi(p_1) = \chi(p_2) \cup a_I$, and $\chi(p_\ell) = \chi(p_{\ell-1}) \cup a_F$, $(P, \chi)$ is a normalized path decomposition of the body of $G$.*

The lemma below is a consequence of the definition of tracks, normalized path decompositions, and $k$-connectivity:

**Lemma 2** *For $G$ a $k$-connected partial $k$-path with at least $k$+3 vertices* `Hole A2: do we want this to refer to the number of body vertices in particular?` *and $(P, \chi, hair)$ any compressed path decomposition of $G$, $P = p_2, \ldots, p_{\ell-1}$,*

1. *$\chi(p_2)$ and $\chi(p_{\ell-1})$ are ends of $G$; and*
2. *for any even $i$, $\chi(p_i)$ contains exactly one vertex from each track.*

**Proof:** We first show that $\chi(p_2)$ is an end of $G$ by showing that $G[V(G) \setminus (\chi(p_2) \cup \text{hair}(p_2))]$ is a single component. Since any bag is a separator, clearly no end can appear in a bag $\chi(p_i)$ for $4 \leq i \leq \ell - 3$.

Suppose instead that there are at least two components in $G[V(G) \setminus (\chi(p_2) \cup \text{hair}(p_2))]$. We let $d = \text{entry}(p_3)$ be a vertex in component $A$ and consider $p_w$, where $w$ is the smallest value such that $d' = \text{entry}(p_w)$ is in another component $B \neq A$. Since $G$ is $k$-connected, there must be $k$ vertex-disjoint paths between $b = \text{exit}(p_3)$ and $d'$. The set $\chi(p_4)$ is a set of size $k$ separating $b$ and $d'$, and hence each of the $k$ paths must contain a distinct member of the set. However, since $\chi(p_2)$ is a set of size $k$ separating $A$ and $B$ (and hence $d$ and $d'$), there cannot be a path from $b$ to $d'$ through $d$ but not through any other member of $\chi(p_2)$, a contradiction. Point 1 for $\chi(p_{\ell-1})$ follows by symmetry.

To prove point 2, it is not difficult to see that for $i = 2$ (and, by symmetry $i = \ell - 1$), $\chi(p_i)$ contains exactly one vertex from each track, as no vertex in the set is replaced by any other vertex in the set). To prove the point for general $i$, we suppose instead that there exists a bag without one vertex from each track, and choose the minimum even $v$ such that $\chi(p_v)$ contains vertices from at most $k - 1$ tracks. We let $t$ be a track such that $\chi(p_v)$ contains no vertex on track $t$, and observe that since $2 < v < \ell - 1$, there exist vertices on track $t$ in bags both preceding and following $(p_v)$. In particular, there must exist a track edge $(a_t, b_t)$ such that $b_t$ replaces $a_t$, $a_t$ appears before $p_v$ and $b_t$ appears after $p_v$. This implies that there exists no bag containing both $a_t$ and $b_t$, although $(a_t, b_t)$ must be an edge and there must be a bag containing both endpoints of any edge, a contradiction. □

We can view a layout of the vertices of $G$ as starting with an initial $k$-leaf at the leftmost point, a final $k$-leaf at the rightmost point, and each track stretched out as a straight line from left to right. Thus, if $b$ replaces $a$ then we say that $a$ is the *track predecessor* of $b$, denoted $\text{pred}(b)$ and that $b$ is the *track successor* of $a$, denoted $\text{succ}(a)$. The *position* of a vertex $a$ on a track, denoted $\text{position}(a)$, is defined as follows: each vertex in the head is in position 1 on its track, and if $b$ replaces $a$, then $\text{position}(b) = \text{position}(a) + 1$. Moreover, for $\text{track}(a) = \text{track}(b)$ and $\text{position}(a) < \text{position}(b)$, $a$ is to the *left* of $b$ and $b$ is to the *right* of $a$. A *track layout* of a graph $G$ of pathwidth $k$ is the numbering of tracks by 1 through $k$ and the association of each vertex $a$ with a pair $(\text{track}(a), \text{position}(a))$.

Our algorithms proceed by attempting to create an embedding by mapping a track layout of $G$ to a track layout of $H$. The mapping will be particularly useful if we select the track layout of $G$ that "corresponds" to the track layout of $H$.

The general idea of the algorithm to find a track layout is as follows: maintain a set $S$ (initially the head vertices, labeled 1 through $k$), and repeatedly find a vertex $a$ of $S$ with one unlabeled neighbor $b$; give $b$ the label of $a$ and have it replace $a$ in $S$. Lemma 3 below implies that the algorithm yields the same layout independent of the order in which vertices are processed. It shows that if two vertices $a_1$ and $a_2$ could both be considered as $a$, they cannot both be replaced by the same vertex $b_1$ (which would lead to two layouts differing in the track label of $b_1$).

**Lemma 3** *Let $G$ be a $k$-connected graph of pathwidth $k$ and $(P', \lambda)$ be a proper prefix of at least one normalized path decomposition of $G$, where $P' = (p_1, \ldots, p_s)$ for $s$ odd. Then if $a_1 \in \lambda(p_s)$, $N(a_1) \setminus \cup_{1 \leq i \leq s} \lambda(p_i) = \{b_1\}$, $a_2 \in \lambda(p_s)$, $N(a_2) \setminus \cup_{1 \leq i \leq s} \lambda(p_i) = \{b_2\}$, and $a_1 \neq a_2$, then $b_1 \neq b_2$.*

**Proof:** For at least one vertex $a \in \chi(p_s)$ such a vertex $b$ exists, namely the vertex $\text{entry}(p_{s+1})$ for some path decomposition $(P, \chi)$ extending $(P', \lambda)$. Suppose instead that for $a_1 \neq a_2$, $b_1 = b_2$. In any normalized path decomposition $(P, \chi)$ of which $(P', \lambda)$ is a prefix, since $(a_1, b_1)$ and $(a_2, b_1)$ are both edges in $G$, there must exist a vertex $p_j$, $j \geq s + 1$, such that $\{a_1, a_2, b_1\} \subseteq \chi(p_j)$ and $\{a_1, a_2, b_1\} \cap \chi(p_{j-1}) = \{a_1, a_2\}$. Then $\chi(p_j) \setminus \{a_1, a_2\}$ is a set of size $k - 1$ separating the initial and final $k$-leaves, violating the $k$-connectivity of $G$. □

Theorem 1 below follows as a consequence. Since either end can be the head, there are at most $2(k!)$ different track layouts of $G$, each of which can be determined in linear time. Arbitrary degree-$k$ neighbors of the head and tail can be identified as the initial and final $k$-leaves, with all other degree-$k$ neighbors being designated as hairs.

**Theorem 1** *For $G$ a $k$-connected graph of pathwidth $k$, for each head $h$ of $G$ and each numbering (permutation) $\pi$ of the tracks, there is a unique track layout $(h, \pi)$ of $G$, which can be generated in linear time.* □

We use $N_t(a)$ to denote the set of neighbors of $a$ on track $t$, and where appropriate we generalize the function to $N_t(A)$ for $A$ a set of vertices.

Although a track layout imposes a total order on the vertices of a particular track, in general it provides only a partial order on the vertices of the entire graph. Given a track decomposition starting from a particular head $h$, $a$ comes before $b$ in the partial order if either $a$ precedes $b$ on the same track, or if there is an edge from $a$ to a vertex to the left of $b$ on the track of $b$. This order reflects the fact that in any path decomposition with head $h$, $a$ must appear in a bag before $b$. The partial order precludes the existence in the track layout of a *transposition*, namely a pair of edges $(a_1, b_2), (a_2, b_1)$ with four distinct endpoints, where $a_i$ $(i = 1, 2)$ is on the same track as and to the left of $b_i$. This is important in proving the correctness of our algorithms.

We can generalize the notion of transposition to provide a forbidden structure useful in proving properties of $k$-connected partial $k$-paths. A *generalized transposition* is a set of edges $\{(a_i, b_{i+1}) | i = 1, 2, \ldots, j\} \cup \{(a_{j+1}, b_1)\}$ for some $1 \leq j \leq k-1$, where $a_i$ $(i = 1, 2, \ldots, j)$ is on the same track as and to the left of $b_i$.

**Lemma 4** *A $k$-connected partial $k$-path cannot contain a generalized transposition.*

**Proof:** The edge $(a_i, b_{i+1})$ shows that $b_{i+1}$ precedes $b_i$ in the partial order (for $i = 1, 2, \ldots j$), but the edge $(a_{j+1}, b_1)$ shows that $b_1$ precedes $b_{j+1}$, a contradiction. □

We can use the preceding lemma to prove that the tracks alone are responsible for a partial $k$-path being $k$-connected.

**Lemma 5** *The set of tracks is the only set of $k$ vertex-disjoint paths from head to tail in a $k$-connected partial $k$-path.*

**Proof:** Suppose there were another set $\mathcal{P}$ of $k$ vertex-disjoint paths from head to tail. Let $P_i$ $(i = 1, 2, \ldots, k)$ be the path from $\mathcal{P}$ passing through the head vertex on track $i$, and let $(c_i, d_i)$ be the first edge of $P_i$ where $\text{track}(c_i) = i$ but $\text{track}(d_i) \neq i$. Since the paths are disjoint, $c_{\text{track}(d_i)}$ must be to the left of $d_i$. Let $F$ be the graph with vertex set $\{(c_i, d_i) | i = 1, 2, \ldots, k\}$ and edge set $\{((c_i, d_i), (c_{\text{track}(d_i)}, d_{\text{track}(d_i)})) | i = 1, 2, \ldots, k\}$. Since $F$ has outdegree one, it must contain a directed cycle, and the edges $\{(c_i, d_i)\}$ which are vertices in this cycle form a generalized transposition, which is a contradiction. □

The next result shows that $k$-connected partial $k$-paths are a rich class of graphs, since the presence of tracks and the obvious lower bound on the degree of a vertex guarantees membership in this class.

**Lemma 6** *Any graph $G$ of minimum degree at least $k$ obtained by starting with a full $k$-path and deleting cross edges is a $k$-connected partial $k$-path.*

**Proof:** Clearly $G$ is a partial $k$-path. Suppose it had a cutset $S$ of size less than $k$. Since $G$ has $k$ tracks, there must be a track not containing a member of $S$, and so the initial and final $k$-leaves of $G$ are in the same component when $S$ is deleted. Let $v$ be a vertex in some other component; it has at least $k-2$ adjacent cross edges. There must be a member of $S$ to the left of $v$ on track $\text{track}(v)$, and one to the right, so there can be at most $k-3$ members of $S$ on tracks different from $\text{track}(v)$, and at least one of the cross edges adjacent to $v$ does not have a member of $S$ as an endpoint.

Suppose there are cross edges between $v$ and $j$ members of $S$, and cross edges between $v$ and vertices not in $S$ on $i \geq 1$ other tracks from the set $T$, where no track in $T$ contains a member of $S$

which is the endpoint of a cross edge. For each of the latter tracks, two members of $S$ must be on that track to disconnect $v$ from the initial and final $k$-leaves. Thus $2i + j \leq k - 3$. Since $v$ has at least $k - 2$ cross edges, there must exist a track $t \in T$ containing three endpoints of cross edges, none of which are in $S$. For $w$ be the middle of the three vertices, $v$ precedes $w$ in the partial ordering, and $w$ is in the same component of $G$ as $v$ when $S$ is deleted.

Since there was nothing special about the choice of $v$, for every vertex in this component we can find a vertex in the same component which it precedes in the partial ordering, which is impossible. □

The preceding two lemmas are not needed in the design or analysis of our algorithms, but give examples of the rich structure associated with $k$-connected partial $k$-paths.

# 4 Topological embedding algorithm

Our algorithm takes an initial injective mapping $f$ of vertices of $G$ to vertices of $H$ and iteratively refines the mapping until it forms an embedding of $G$ into $H$ or fails. Throughout the execution of the algorithm, the possible mappings considered will be constrained by track layouts of $G$ and $H$. By choosing an arbitrary path decomposition of $H$, we can fix a total order on the vertices of $H$ and a specific track layout. If $G$ is embeddable in $H$, one of the $2(k!)$ track layouts of $G$ will be associated with the path decomposition of $G$ induced by the embedding. We use Figure 4 to illustrate a possible embedding of $G$ (Figure 2.1) in $H$ (Figure 2.1); the $k$-bags in the induced path decomposition of $G$ are extracted in two steps, depicted in Tables 4 and 4 below.

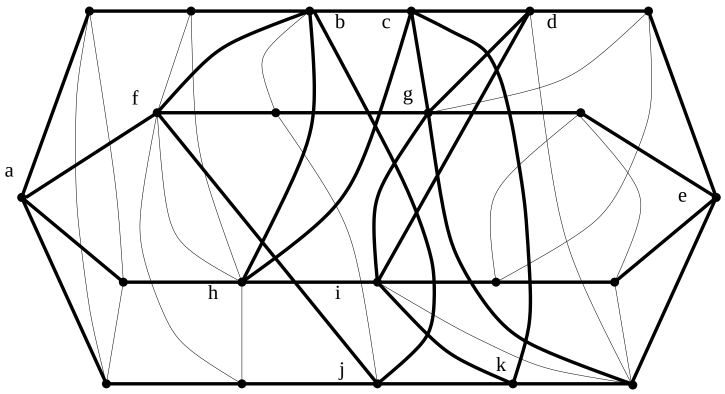


Figure 5: A topological embedding of $G$ in $H$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $b$ | $b$ | $b$ | $c$ | $c$ | $c$ | $c$ | $c$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ |
| $a$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $j$ | $j$ | $j$ | $j$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ |

Table 4: Mapping bags under the topological embedding

We initially restrict our focus to $G$ and $H$ $k$-connected partial proper $k$-paths, and then we discuss extensions to more general situations. The track layouts of $G$ and $H$ can be exploited in the discussion of an embedding of $G$ into $H$; Lemma 8 below shows that track numbers and cross edges are preserved under the embeddings. A topological embedding $(f, \phi)$ from $G$ into $H$ that satisfies the conditions in the lemma is said to be a *topological embedding with respect to layouts* $(h_G, \pi_G)$ *and* $(h_H, \pi_H)$. Lemma 8 can be proved by showing that the violation of any of the conditions of the lemma allows

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $b$ | $b$ | $c$ | $c$ | $c$ | $c$ | $c$ | $d$ |
| $f$ | $f$ | $f$ | $f$ | $f$ | $g$ | $g$ | $g$ |
| $h$ | $h$ | $h$ | $i$ | $i$ | $i$ | $i$ | $i$ |
| $j$ | $j$ | $j$ | $j$ | $k$ | $k$ | $k$ | $k$ |

Table 5: Induced path decomposition of $G$ under the topological embedding

us, by using induced path decompositions, to find a bag in the path decomposition of $G$ violating Lemma 2, point 2.

**Lemma 7** *For $(f, \phi)$ a topological embedding of $G$ into $H$ and $(P, \chi)$ any path decomposition of $H$, for each end $E$ of $G$, there must exist $p_r$, $r$ even, such that $\chi(p_r)$ contains the images of each vertex in $E$.*

**Proof:** If there were no such bag $p_r$, then the induced path decomposition of $G$ would contain no bag consisting of the vertices in $E$. Since by Lemma 1 the induced path decomposition of $G$ is a normalized path decomposition, it must contain such a bag (Lemma 2, point 1), a contradiction. □

**Lemma 8** *For $G$ and $H$ $k$-connected partial proper $k$-paths, $G$ is topologically embeddable in $H$ if and only if there exists a topological embedding $(f, \phi)$ and track layouts $(h_G, \pi_G)$ of $G$ and $(h_H, \pi_H)$ of $H$ such that for $a_I$ and $a_F$ the initial and final $k$-leaves of $G$ and $u_I$ and $u_F$ the initial and final $k$-leaves of $H$:*

1. *for all $a \in V(G) \setminus \{a_I, a_F\}$, track$(a)$ = track$(f(a))$;*
2. *for each track edge $(a, b)$ in $G$, $a \neq a_I$ and $b \neq a_F$, $\phi((a, b))$ consists of the path from $f(a)$ to $f(b)$ on the track of $a$;*
3. *for each cross edge $(a, b)$ in $G$, $\phi((a, b))$ consists of the edge $(f(a), f(b))$; and*
4. *$f(a_I) = u_I$, $f(a_F) = u_F$, for all edges $(a_I, b)$, $\phi((a_I, b))$ is a path from $f(a_I)$ to $f(b)$ with all interior vertices on the track of $b$, and for all edges $(b, a_F)$, $\phi((b, a_F))$ is a path from $f(b)$ to $f(a_F)$ with all interior vertices on the track of $b$.*

**Proof:** Suppose $(f, \phi)$ is a topological embedding of $G$ into $H$, $(P, \chi)$ a path decomposition of $H$ and $(P_G, \chi_G)$ the path decomposition of $G$ induced by $(f, \phi)$. We can clearly fix a track layout of $H$ and create a layout for $G$ dictated by the tracks in $H$ on which each vertex of $G$ is found.

We prove conditions 1 through 3 by supposing instead they were not all true, and finding the first place (based on the total ordering of vertices in $H$) at which one of the conditions is violated. For each of the three cases we obtain a contradiction.

**Case 1:** Condition 1 is violated first.

Suppose that the first violation occurs when there is an $a \in V(G)$ such that track$(f(a)) \neq$ track$(a)$. If $a$ is a neighbor of the initial $k$-leaf $a_I$ then a violation occurs at $a$ only if the neighbors of $a_I$ map to fewer than $k$ different tracks. Since by Lemma 2, point 1 the neighborhood of $a_I$ forms a bag in $(P_G, \chi_G)$, this violates Lemma 2, point 2.

If instead $a$ is not a neighbor of $a_I$, we let $v$ be the predecessor of $f(a)$ on the path in $H$ corresponding to the edge $(\text{pred}(a), a)$ in $G$; $v$ is in the attachment clique of $f(a)$. Since $v$ adheres to the conditions, track$(v)$ = track$(a)$. We now consider the vertex $p_i \in P$, $i$ odd, in which $f(a)$ first appears; both pred$(f(a))$ and $v$ are in $\chi(p_i)$. By properties of tracks pred$(f(a)) \notin \chi(p_{i+1})$, and hence both $f(a)$ and $v$ are in $\chi(p_{i+1})$. Since $\psi(v) \in \{\text{pred}(a), a\}$, $\psi(f(a)) = a$, and track(pred$(a)$) = track$(a)$, in the path decomposition of $G$ induced by $(f, \phi)$, $\chi_G(p_{i+1})$ contains vertices on at most $k-1$ different tracks, violating Lemma 2, point 2.

**Case 2:** Condition 2 is violated first.

The argument is similar to that given in Case 1; in this case we suppose that the first violation occurs when there is a vertex on the path $\phi((a,b))$ which is not on track track$(a)$. We know that $f(a)$ is on track$(a)$, but $f(b)$ is not necessarily on track track$(b)$. We can then conclude that there is an edge in $\phi((a,b))$ between a vertex $v$ on track$(f(b))$ and a vertex $w$ on a different track. As in the previous case, we consider the first vertex $p_i$ in which $v$ appears. The vertex $w$ is in the attachment clique of $v$, and hence both $w$ and pred$(v)$ are in $\chi(p_i)$. By properties of tracks pred$(v) \notin \chi(p_{i+1})$, and hence both $w$ and $v$ are in $\chi(p_{i+1})$. Since $\psi(v)$ and $\psi(w)$ are either $a$ or $b$, in the path decomposition of $G$ induced by $(f, \phi)$, $\chi_G(p_{i+1})$ contains vertices on at most $k-1$ different tracks, violating Lemma 2, point 2.

**Case 3:** Condition 3 is violated first.

By our assumption, $\phi(f(a))$ does not consist of the edge $(f(a), f(b))$. Since this is the first violation of a condition, we know that $f(a)$ is on track$(a)$ and that $\phi((\text{pred}(a), a))$ is a path from $f(\text{pred}(a))$ to $f(a)$ on track track$(a)$. If the path $\phi((a,b))$ includes the track edge from $f(a)$ to succ$(f(a))$, then Condition 2 is violated for the track edge $(a, \text{succ}(a))$, reducing to Case 2. We can then conclude that either $\phi((a,b))$ contains track edges on track track$(f(b))$ (violating Condition 2 for either $(\text{pred}(b), b)$ or $(b, \text{succ}(b))$) or $\phi((a,b))$ contains an edge from a vertex $w$ on track $t$, $t \neq \text{track}(a)$, $t \neq \text{track}(b)$, to $f(b)$.

We observe that there must be a bag in the path decomposition containing $f(b)$ and $w$. Since $\psi(w) \in \{\text{track}(a), \text{track}(b)\}$, the induced path decomposition of $G$ will contain a bag without any vertex on track $t$, violating Lemma 2, condition 2.

We now consider what vertices can be found on $\phi((a,b))$ We assume that and that $f(\text{pred}(b))$ is on track$(b)$. If $f(b)$ is not on track track$(b)$, then condition 2 is violated for the edge $(\text{pred}(b), b)$, reducing to Case 2.

In this case, either there is a vertex on the path $\phi((a,b))$ which is not on track track$(a)$ and is not $f(b)$, or there is a vertex on the path $\phi((a,b))$ which is not on track track$(b)$ and is not $f(a)$. In

$u \neq f(a)$, $u \neq f(b)$, but track$(a) \neq$ track$(b)$. We can find $v$ and $w$ on the path from $f(a)$ to $f(b)$ as in Case 2, and the remainder of the proof is identical.

To prove condition 4, we observe that as a consequence of condition 1 and Lemma 2 and Lemma 7, each vertex in an end can be found on a distinct track. Moreover, since the images of the end vertices appear in a bag (and hence separate $f(a_I)$ or $f(a_F)$ from the rest of the graph) we can form the paths as indicated in the statement of the condition. Finally, the proof of the reverse implication is obvious. □

The algorithm starts by forming a single track layout and total order for $H$ and all $2(k!)$ track layouts of $G$. For each possible layout of $G$, initially we assign $f(a) := u$, where track$(a) =$ track$(u)$ and position$(a) =$ position$(u)$. Given a mapping of vertices of $G$ to vertices of $H$, extended in the obvious way to map sets of vertices, we say that $a \in V(G)$ is *consistent* if for all tracks $t$, $f(N_t(a)) \subseteq N_t(f(a))$. We then repeatedly check consistency of vertices in $G$, resolving any inconsistency by "sliding" vertices to later positions on their track. An inconsistency in which $(a,b) \in E(G)$ but $(f(a), f(b)) \notin E(H)$ is resolved by changing one or both of $f(a)$ and $f(b)$. Since there are no transpositions in $(h_H, \pi_H)$, the total order on vertices in $H$ induces a total order on edges between any two tracks in the layout of $H$.

**Definition 8** *The* leftmost consistent edge *with respect to $a, b \in G$ and mapping $f$ is the leftmost edge $(u,v)$ under the total ordering of edges in $H$ between the track of $f(a)$ and $f(b)$ such that position$(f(a)) \leq$ position$(u)$ and position$(f(b)) \leq$ position$(v)$.*

To ensure $(f(a), f(b)) \in E(H)$, $f(a)$ is set to $u$ and $f(b)$ is set to $v$ (which can be viewed as "sliding" $f(a)$ or $f(b)$ along its track), and we update function $f$ by "sliding" vertices to the right of $a$ and $b$. Either all the inconsistencies are resolved or we fail by attempting to slide a vertex off the end of its track.

```
**set up data structures**
form track layout and total order of H
for each possible layout of G
   set up data structure for G
   **initial assignment of G to H**
   for each a ∈ V(G)
      f(a) := (track(a), position(a)) in H
      (if no such vertex, stop and declare failure)
   end for
   put all vertices in I
   repeat until I is empty:
      extract a from I
      check(a)
```

The subroutine check examines a vertex $a$ to see if for any edge $(a, b)$ in $G$, the edge $(f(a), f(b))$ does not exist in $H$. If not, an inconsistency has been detected, and the positions of $a$ and $b$ in the mapping $f$ must be adjusted. The adjustment is made by finding the leftmost consistent edges $(u, v)$ and then setting $f(a)$ to $u$ and $f(b)$ to $v$, also adjusting vertices to the right of $a$ and $b$. The way in which $a$ is chosen from $\mathcal{I}$ and neighbors $b$ of $a$ are checked turns out to not affect the correctness or asymptotic complexity of the algorithm, as long as the set operations take constant time.

```
procedure(check(a))
if (f(a), f(b)) ∉ E(H) for any (a, b) ∈ E(G)
   let (u,v) be the leftmost consistent edge w.r.t.  a, b, f
   (if no such edge, stop and declare failure)
   for all c with track(c) = track(a), position(c) > position(a)
   ** make room for move on track(a) **
      if position(u) + position(c) − position(a) > position(f(c))
         f(c) := (track(c), position(u)+position(c)-position(a))
         (if no such position, stop and declare failure)
         add c to I
      end if
   end for
   ** make room for move on track(b) **
   for all d with track(d) = track(b), position(d) > position(b)
      if position(v) + position(d) − position(b) > position(f(d))
         f(d) := (track(d), position(v)+position(d)-position(b))
         (if no such position, stop and declare failure)
         add d to I
      end if
   end for
   ** move a,b **
   add a, b to I
```

```
    f(a) := u
    f(b) := v
end if
```

**Lemma 9** *For $G$ and $H$ $k$-connected partial proper $k$-paths, if there exist track layouts $(h_G, \phi_G)$ of $G$ and $(h_H, \phi_H)$ of $H$, $a_I$ and $a_F$ the initial and final $k$-leaves of $G$, $u_I$ and $u_F$ the initial and final $k$-leaves of $H$, and a mapping satisfying following conditions, then there is a topological embedding from $G$ into $H$.*

*i. for each $a \in V(G) \setminus \{a_I, a_F\}$, track$(a)$ = track$(f(a))$;*

*ii. for vertices $a$ and $b$ in $V(G)$ such that track$(a)$ = track$(b)$, if position$(a)$ < position$(b)$, then position$(f(a))$ < position$(f(b))$;*

*iii. for each $a \in V(G)$, $a$ is consistent; and*

*iv. $f(a_I) = \{u_I\}$, $f(a_F) = \{u_F\}$.*

**Proof:** Given a mapping $f$ which satisfies all four conditions, we claim that $(f, \phi)$ is a topological embedding of $G$ into $H$ with respect to layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$, where $\phi$ maps each edge as in the statement of Lemma 8. That is, $\phi((a, b))$ is the edge $(f(a), f(b))$ (whose existence is guaranteed by (iii)) if $(a, b)$ is a cross edge in $G$, $\phi((a_I, b))$ is a path from $f(a_I)$ to $f(b)$ with all interior vertices on the track of $b$ for all edges $(a_I, b)$ in $G$, and $\phi((b, a_F))$ is a path from $f(b)$ to $f(a_F)$ with all interior vertices on the track of $b$ for all edges $(b, a_F)$.

Clearly $f$ and $\phi$ are both functions as described in the definition of topological embedding (Definition 3). Condition 1 in that definition is satisfied by the definition of $\phi$ above. Condition (ii) above implies that no track edge in $H$ is used in the image under $\phi$ of more than one track edge of $G$, hence all images under $\phi$ are vertex-disjoint except possibly at the end vertices, as required to satisfy condition 2 of Definition 3. □

Given $G$ topologically embeddable in $H$ with respect to track layouts $(h_G, \pi_G)$ of $G$ and $(h_H, \pi_H)$ of $H$, we can determine a partial order among topological embeddings associated with the track layouts, where $(f_1, \phi_1)$ comes before $(f_2, \phi_2)$ if for all $a \in V(G)$, $\text{position}(f_1(a)) \leq \text{position}(f_2(a))$. Our algorithm finds the unique minimum embedding under this partial order, whose existence is guaranteed by the following lemma.

**Lemma 10** *If $G$ is topologically embeddable in $H$ with respect to $(h_G, \pi_G)$ and $(h_H, \pi_H)$, then there is a unique minimum $f_m$ (with $\phi$ defined as in Lemma 8) associated with $(h_G, \pi_G)$ and $(h_H, \pi_H)$.*

**Proof:** Suppose instead there were incomparable minimal mappings $(f_1, \phi_1)$ and $(f_2, \phi_2)$; there must exist $a$ and $b$ in $V(G)$ such that the following conditions all hold: $\text{track}(a) \neq \text{track}(b)$, $\text{position}(f_1(a)) < \text{position}(f_2(a))$ and $\text{position}(f_2(b)) < \text{position}(f_1(b))$. We can partition the vertices in $V(G)$ into the following three sets: $S_1 = \{a \in V(G) \mid \text{position}(f_1(a)) < \text{position}(f_2(a))\}$, $S_2 = \{a \in V(G) \mid \text{position}(f_2(a)) < \text{position}(f_1(a))\}$, and $S_= = \{a \in V(G) \mid \text{position}(f_1(a)) = \text{position}(f_2(a))\}$.

We observe that there cannot exist an edge in $E(G)$ between $a \in S_1$ and $b \in S_2$, since the edges $(f_1(a), f_1(b))$ and $(f_2(a), f_2(b))$ form a transposition in $H$. Consequently, all edges are either between vertices in the same set, between $S_1$ and $S_=$, or between $S_2$ and $S_=$.

We can form $f_3$ such that for $a \in S_1$, $f_3(a) = f_1(a)$, for $a \in S_2$, $f_3(a) = f_2(a)$, and for $a \in S_=$, $f_3(a) = f_1(a) = f_2(a)$. Clearly all edges can be mapped, and hence $f_3$ is an embedding violating the minimality of $f_1$ and $f_2$, yielding a contradiction. □

**Lemma 11** *If $G$ is topologically embeddable in $H$ with respect to track layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$, then at any point during the execution of the algorithm above where these track layouts are chosen, and for any vertex $a \in V(G)$, position$(f(a)) \leq$ position$(f_m(a))$, where $f_m$ is the unique minimal $f$ (as defined in Lemma 10).*

**Proof:** Suppose not. Consider the first moment at which the property is violated. By symmetry, there are two cases: either a vertex $a$ was moved as a result of an inconsistency involving $a$ and $b$, or a vertex $c$ was moved as a result of an inconsistency involving a vertex $a$ earlier on the track and a vertex $b$ on some different track ($a$ was required to move and hence pushed $c$ to the right).

In the first case, the edge $(f(a), f(b))$ did not exist in $H$, but the edge $(u, v)$ did. The violation of the property means that position$(f_m(a)) <$ position$(u)$. Since the edge $(f_m(a), f_m(b))$ must also exist in $H$, the lack of transpositions implies that position$(f_m(b)) \leq$ position$(v)$. But then $(f_m(a), f_m(b))$ should have been chosen instead of $(u, v)$.

In the second case, note that the condition under which $c$ is moved means that all vertices between $a$ and $c$ map to consecutive positions on the same track in $H$. This means that when $a$ is moved, it also must violate the property, and this reduces to the first case. □

**Theorem 2** *For $G$ and $H$ $k$-connected partial proper $k$-paths, it is possible to determine whether or not $G$ is topologically embeddable in $H$ in $O(n^3)$ time.* □

**Proof:** We have shown (Lemma 11) that if $G$ is topologically embeddable in $H$ with respect to given track layouts, then our algorithm never pushes vertices past their positions in the unique minimal mapping (Lemma 10). Since the algorithm terminates when all vertices are consistent, it will terminate when each vertex is at the position dictated by the minimal mapping. Finally, we observe that if our algorithm terminates by finding that all vertices are consistent, then all conditions of Lemma 9 have been satisfied and hence there exists a topological embedding from $G$ to $H$.

The algorithm of Section 3 finds track layouts in linear time. In the topological embedding algorithm, each vertex slides forward at most $n$ positions for a total of at most $O(n^2)$ slides. Each slide is the consequence of a failed consistency check. To check all edges takes $O(n)$ time, and so in $O(n)$ time an inconsistent pair can be detected, if one exists. The work done in each slide is $O(n)$. Thus we have described an $O(n^3)$ algorithm for topological embedding of $k$-connected partial proper $k$-paths. □

When $H$ can have hairs and thus is no longer proper, the situation is more complicated. A path of two hair edges in $H$ can be used to embed a cross edge of $G$ in the preimage of the attachment clique of that hair (which may no longer be a clique, since $G$ is partial). Since there may be more than one candidate cross edge, ambiguity is introduced. We are able to resolve this ambiguity to obtain $O(n^3)$ algorithms when one of $G$ and $H$ is a full $k$-path and the other a $k$-connected partial $k$-path.

When $H$ is a full $k$-path, all edges between vertices in the attachment clique of a hair are present, and can be used for embedding cross edges. This means that if $G$ is topologically embeddable in $H$ there exists a embedding in which the only use of hairs in $H$ is to map hairs in $G$ onto them.

To adapt the algorithm for $G$ a $k$-connected partial $k$-path and $H$ a full $k$-path, we simply run the algorithm for the bodies of both $G$ and $H$. When the set of possibly inconsistent vertices $\mathcal{I}$ is empty, a mapping from the body of $G$ to the body of $H$ has been found, and the algorithm proceeds to check whether, for each attachment clique $C$ of $G$ with $h$ hairs, there are at least $h$ hairs attached to $f(C)$. If not, the vertices of $C$ slide forward to the next attachment clique of $H$, they are all added to $\mathcal{I}$, and the algorithm is restarted from the point at which it stopped. This does not increase the asymptotic complexity of the algorithm.

**Theorem 3** *For $G$ a $k$-connected partial $k$-path and $H$ a full $k$-path, it is possible to determine whether or not $G$ is topologically embeddable in $H$ in $O(n^3)$ time.* □

`Hole 6: add details`

Similarly, if $G$ is a full $k$-path and $H$ a partial $k$-path, every cross edge in $G$ is part of an actual clique. If the subgraph of $H$ induced by the image of this clique is missing any cross edges, the corresponding edges of $G$ must be mapped to hairs attached to that image. Again, the asymptotic complexity of the algorithm does not increase.

**Theorem 4** *For $G$ a full $k$-path and $H$ a $k$-connected partial $k$-path, it is possible to determine whether or not $G$ is topologically embeddable in $H$ in $O(n^3)$ time.* □

`Hole 7: add details`

# 5 Minor embedding algorithm

In the case of minor embedding, it would seem that vertices of $G$ may now map to seemingly arbitrary connected subgraphs of $H$. However, as we will see, Lemma 14 gives a structural characterization which limits possible images of vertices. Throughout this section we assume that $G$ and $H$ are $k$-connected partial proper $k$-paths. To illustrate our algorithm, we consider a minor embedding of $G$ (Figure 2.1) in $H$ (Figure 2.1) in Figure 5; the induced path decomposition is derived in Tables 5 and 5.

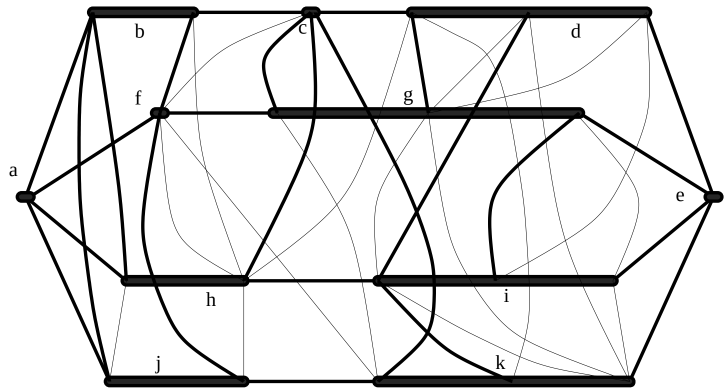


Figure 6: A minor embedding of $G$ in $H$

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $b$ | $b$ | $b$ | $b$ | $c$ | $c$ | $c$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| $f$ | $f$ | $f$ | $f$ | $f$ | $f$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ |
| $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ | $i$ |
| $j$ | $j$ | $j$ | $j$ | $j$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ | $k$ |

Table 6: Mapping bags under the minor embedding

| | | | | | |
|---|---|---|---|---|---|
| $b$ | $c$ | $c$ | $c$ | $d$ | $d$ |
| $f$ | $f$ | $f$ | $g$ | $g$ | $g$ |
| $h$ | $h$ | $h$ | $h$ | $h$ | $i$ |
| $j$ | $j$ | $k$ | $k$ | $k$ | $k$ |

Table 7: Induced path decomposition of $G$ under the minor embedding

To facilitate the proof of Lemma 14, we first establish a few properties of minor embeddings. We focus first on the role of the initial and final $k$-leaves. Recall that the minor embedding function $f$ maps vertices of $G$ to connected subgraphs of $H$.

**Lemma 12** *For any minor embedding $(f, \xi)$ of $G$ into $H$, for any $a \in V(G)$, if $f(a)$ does not contain either the initial or final $k$-leaf of $H$, then for any track layout $(h_H, \pi_H)$ of $H$ there exists a track $t$ in $(h_H, \pi_H)$ such that $f(a)$ consists of an interval of vertices $[u, v]$ on $t$.*

**Proof:**

We first demonstrate that $f(a)$ cannot contain any cross edge $(v, w)$ of $H$. Since $v$ and $w$ are neighbors but neither is the exit vertex of the other, in any path decomposition $(P, \chi)$ of $H$ there must exist a bag $p_r$, $r$ even, such that $\{v, w\} \subseteq \chi(p_r)$. If $v$ and $w$ are in $f(a)$, then in the induced path decomposition $(P_G, \chi_G)$, $|\chi_G(p_r)| < k$, contradicting the $k$-connectivity of $G$.

Since $f(a)$ forms a connected subgraph of $H$ and contains no cross edge nor the initial nor final $k$-leaf, $f(a)$ is an interval of vertices $[u, v]$ on a single track $t$ in $H$. □

The proof of the following lemma is identical to that of Lemma 7.

**Lemma 13** *For $(f, \xi)$ a minor embedding of $G$ into $H$ and $(P, \chi)$ be any path decomposition of $H$, for each end $E$ of $G$ there must exist $p_r$, $r$ even, such that $\chi(p_r)$ contains the images of each vertex in $E$* □

With the aid of the preceding two lemmas, the proof of Lemma 14 is similar to, though more complicated than, the proof of Lemma 8 (the analogous lemma for topological embedding).

**Lemma 14** *For $G$ and $H$ $k$-connected partial proper $k$-paths, $G$ is a minor of $H$ if and only if there exist a minor embedding $(f, \xi)$ and track layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$ such that, for $a_I$ and $a_F$ the initial and final $k$-leaves of $(h_G, \pi_G)$ and $u_I$ and $u_F$ the initial and final $k$-leaves of $(h_H, \pi_H)$, the following conditions hold:*

1. *for all $a \in V(G) \setminus \{a_I, a_F\}$, $f(a)$ is an interval of vertices $[\ell(f(a)), r(f(a))] = [u, v]$ on the track of $a$;*
2. *for each track edge $(a, b)$ in $G$, $\xi((a, b))$ consists of the edge from $r(f(a))$ to $\ell(f(b))$;*
3. *for each cross edge $(a, b)$ in $G$, $\xi((a, b))$ consists of an edge from a vertex in $f(a)$ to a vertex in $f(b)$; and*
4. *$f(a_I) = \{u_I\}$ and $f(a_F) = \{u_F\}$.*

**Proof:** To prove the nontrivial direction, since $G$ is a minor of $H$, there must exist a minor embedding $(g, \gamma)$ from $G$ to $H$. Our proof proceeds by altering this embedding to form $(f, \xi)$ and choosing track layouts which satisfy the conditions. We fix $(h_H, \pi_H)$ and then create $(f, \xi)$ and a layout for $G$ dictated by the tracks in $H$ on which each vertex of $G$ is found. For $a \in \{a_I, a_F\}$, we let $b_1, \ldots, b_k$ be the neighbors of $a$. Since any bag in a path decomposition is a separator in the graph, as a consequence of Lemma 13 we can conclude that the $f(b_j)$'s separate $f(a)$ from the remainder of the graph. To satisfy condition 4 it will suffice to consider the mapping of $a$ and its neighbors to the prefix or suffix of $(h_H, \pi_H)$ up to and including the $f(b_j)$'s.

By Lemmas 12, 13, and 2, point 2, each $f(b_j)$ maps to a distinct track in $(h_H, \pi_H)$. By choosing $(h_G, \pi_G)$ such that $\text{track}(b_j) = \text{track}(f(b_j))$, we satisfy condition 1 for each $b_j$.

We consider two cases for each $a \in \{a_I, a_F\}$, depending on whether or not $g(a)$ contains a $u \in \{u_I, u_F\}$.

**Case 1:** $g(a)$ contains $u$

We can direct $(h_G, \pi_G)$ so that $u = u_I$ if and only if $a = a_I$. We then form $f(a)$ by restricting $g(a)$ to the single vertex $u$, and then set each $f(b_j)$ to be the union of $g(b_j)$ and each vertex on the path from $u$ to the leftmost vertex in $g(b_j)$ (that is, an initial segment of the vertices on track track$(b_j)$).

**Case 2:** $g(a)$ does not contain $u_I$ or $u_F$

We can conclude from Lemma 12 that $g(a)$ consists of an interval on a track $t$ in $H$. By condition 1 applied to the $b_j$'s we can conclude that there is a path from $g(a)$ to $u \in \{u_I, u_F\}$ which contains no vertex in $g(b_j)$ for $1 \leq j \leq k$. We choose $(h_G, \pi_G)$ so that $u = u_I$ if and only if $a = a_I$.

To create $(f, \xi)$, we set $f(a) = u$. We can then extend the images of the $b_j$'s in order to include the paths from $u$ to each $g(b_j)$. In particular, for $b_r$ such that track$(g(b_r))$ = track$(g(a))$, we set $f(b_r)$ to be the union of $g(b_r)$, $g(a)$, and the path from $g(a)$ to $u$ on track$(g(a))$. For each other $b_j$ such that $j \neq r$, we set $f(b_j)$ to be the union of $g(b_j)$ and the path from $g(b_j)$ to $u$ on track$(g(b_j))$.

In both cases we have satisfied condition 4 and in addition conditions 1 and 2 for the two $k$-leaves and their neighbors. To see that condition 1 holds for the remaining vertices, we use an argument similar to that used to prove Lemma 8 in a proof by induction on track position. Namely, we show that if a track successor $c$ of a $b_j$ is mapped to a track other than track$(b_j)$, then there exists a bag in a path decomposition of $H$ containing vertices in both $f(b_j)$ and $f(c)$. The result follows from the fact that this will violate Lemma 2, point 2 in the induced path decomposition of $G$.

To complete the proof, we observe that condition 3 follows from the fact that $(g, \gamma)$ is a minor embedding, and that we can define $(f, \xi)$ to satisfy condition 2, given condition 1. □

We modify the algorithm for topological embedding to obtain an $O(n^3)$ algorithm for determining, for $k$-connected partial proper $k$-paths $G$ and $H$, if $G$ is a minor of $H$. We establish a tentative minor embedding and then check for inconsistencies. Initially for $a \in V(G)$, the tentative minor embedding assigns $f(a)$ to be the single vertex at (track$(a)$, position$(a)$).

As in the topological embedding algorithm, resolving an inconsistency entails not only "pushing" the image of one or more endpoints of an edge in $G$ to line up with an edge in $H$, but also finding new assignments of track successors to reestablish conditions 1 and 2 in Lemma 9. Suppose there is an edge $(a, b)$ in $G$ such that there is no edge from the subgraph $f(a)$ to the subgraph $f(b)$ in $H$. Since conditions 1 and 2 hold, track$(f(a)) \neq$ track$(f(b))$. We generalize the leftmost consistent edge, as defined in Section 4, to be the leftmost edge such that position$(u) \geq$ position$(r(f(a)))$ and position$(v) \geq$ position$(r(f(b)))$. For $(u, v)$ the leftmost consistent edge, we assign $r(f(a))$ to be $u$ and $r(f(b))$ to be $v$.

Since the new $f(a)$ and $f(b)$ may now overlap other intervals, we must "push" such intervals further down their tracks as needed. For each $c \in V(G)$ such that track$(f(c))$ = track$(f(a))$, $r(f(c))$ is defined to be the rightmost of its current value and the vertex in position position$(r(f(a)))$ + position$(c)$ − position$(a)$.

To show that the algorithm works correctly, we prove analogues of lemmas proved for topological embedding, with Lemma 15 corresponding to Lemma 9, Lemma 16 corresponding to Lemma 10, and Lemma 17 corresponding to Lemma 11.

**Lemma 15** *For $G$ and $H$ $k$-connected partial proper $k$-paths, if there exist track layouts $(h_G, \phi_G)$ of $G$ and $(h_H, \phi_H)$ of $H$, $a_I$ and $a_F$ the initial and final $k$-leaves of $G$, $u_I$ and $u_F$ the initial and final $k$-leaves of $H$, and a mapping satisfying following conditions, then $G$ is a minor of $H$.*

*i. for each vertex $a \in V(G) \setminus \{a_I, a_F\}$, $f(a) = [u, v]$ for some $u$ and $v$ on the track of $a$ in $H$;*

*ii. for each vertex $a \in V(G)$ with track successor $b$, position$(\ell(f(b))) = 1 +$ position$(r(f(a)))$;*

*iii. for each $a \in V(G)$, $a$ is consistent; and* *iv.* $f(a_I) = \{u_I\}$*;* $f(a_F) = \{u_F\}$*.*

**Proof:** Given a mapping $f$ which satisfies all four conditions, we claim that $(f, \xi)$ is a minor embedding of $G$ into $H$ with respect to layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$, where $\xi$ maps each edge as in the statement of Lemma 14. Namely, $\xi((a, b))$ consists of the edge from $r(f(a))$ to $\ell(f(b))$ (which exists by (ii) above) for each track edge $(a, b)$ in $G$, and of an edge from a vertex in $f(a)$ to a vertex in $f(b)$ for each cross edge $(a, b)$ in $G$ (which exists by (iii) above).

To see that Conditions 1–4 in the definition of minor embedding (Definition 5) are satisfied, we first observe that Condition 1 follows from Condition (i) above, and Condition 4 follows from Condition (ii) above. Condition 3 follows from the definition of $\xi$, and Condition 2 follows from that definition and Condition (ii). □

We can determine a partial order among minor embeddings associated with track layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$, where $f_1$ comes before $f_2$ if for all $a \in G$, position$(r(f_1(a))) \leq$ position$(r(f_2(a)))$.

**Lemma 16** *If $G$ is a minor of $H$ with respect to $(h_G, \pi_G)$ and $(h_H, \pi_H)$, then there is a unique minimum $f_m$ (with $\xi$ defined as in Lemma 14) associated with $(h_G, \pi_G)$ and $(h_H, \pi_H)$.*

**Proof:** Suppose instead there were incomparable minimal mappings $f_1$ and $f_2$. We claim that we can then form a smaller mapping $f_3$ from $f_1$ and $f_2$, contradicting the minimality of $f_1$ and $f_2$.

For each $a$ in $V(G)$ we let $r(f_3(a))$ be the leftmost of $r(f_1(a))$ and $r(f_2(a))$. As $\ell(f_3(a))$ will be determined by the placement of the track predecessor of $a$, it follows that at least one of $f_1(a)$ and $f_2(a)$ is entirely contained in $f_3(a)$. Moreover, any vertex in $f_1(a) \backslash f_3(a)$ or $f_2(a) \backslash f_3(a)$ must fall to the right of $f_3(a)$.

To complete the proof of the lemma it will suffice to show that for each edge $(a, b)$ in $E(G)$ there is an edge from $f_3(a)$ to $f_3(b)$. Suppose instead there were an edge $(u, v)$ in $E(G)$ such that there were no edge from $f_3(a)$ to $f_3(b)$; $(u, v)$ must be a cross edge.

Without loss of generality, we can assume that $f_1(a)$ is contained in $f_3(a)$. It then follows that $f_2(b)$ but not $f_1(b)$ is contained in $f_3(b)$, since otherwise the edge from $f_1(a)$ to $f_1(b)$ would be an edge from $f_3(a)$ to $f_3(b)$.

We now consider the edges $e_1$ from $f_1(a)$ to $f_1(b)$ and $e_2$ from $f_2(a)$ to $f_2(b)$, which must exist since $f_1$ and $f_2$ are both minor embeddings. Since there is no edge from $f_3(a)$ to $f_3(b)$, and one endpoint of $e_1$ is in $f_1(a)$ (which is in $f_3(a)$), the other endpoint must be in $f_1(b) \backslash f_3(b)$, that is, to the right of $f_3(b)$. Similarly, $e_2$ must connect a node in $f_3(b)$ to a node in $f_2(a) \backslash f_3(a)$. Then $e_1$ and $e_2$ form a transposition, contradicting the fact that $(h_H, \pi_H)$ is a track layout of $H$. □

**Lemma 17** *If $G$ is a minor of $H$ with respect to track layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$, then at any point during the execution of the algorithm above where these track layouts are chosen, and for any vertex $a \in V(G)$, position$(r(f(a)) \leq$ position$(r(f_m(a))$, where $f_m$ is the unique minimal $f$ (as defined in Lemma 16).*

**Proof:** Suppose not. Consider the first moment at which the property is violated. By symmetry, there are two cases: either a vertex $a$ was moved as a result of an inconsistency involving $a$ and $b$, or a vertex $c$ was moved as a result of an inconsistency involving a vertex $a$ earlier on the track and a vertex $b$ on some different track ($a$ was required to move and hence pushed $c$ to the right).

In the first case, there was no edge from a vertex in $f(a)$ to a vertex in $f(b)$, and $(u, v)$ is the leftmost consistent edge in $H$. The violation of the property means that position$(r(f_m(a)) <$ position$(u)$. Since there must exist in $H$ an edge joining vertices in $f_m(a)$ and $f_m(b)$, say $e = (u_a, u_b)$, since $u_a \leq f_m(a)$, the lack of transpositions implies that position$(u_b) \leq$ position$(v)$. But then $e$ should have been chosen instead of $(u, v)$.

In the second case, note that the condition under which $c$ is moved means that all vertices between $a$ and $c$ map to consecutive positions on the same track in $H$. This means that when $a$ is moved, it also must violate the property, and this reduces to the first case. □

**Theorem 5** *For $G$ and $H$ $k$-connected partial proper $k$-paths, it is possible to determine whether or not $G$ is a minor of $H$ in $O(n^3)$ time.* □

**Proof:** We have shown (Lemma 17) that if $G$ is a minor of $H$ with respect to given track layouts, then our algorithm never pushes vertices past their positions in the unique minimal mapping (Lemma 16). Since the algorithm terminates when all vertices are consistent, it will terminate when each vertex is at the position dictated by the minimal mapping. Finally, we observe that if our algorithm terminates by finding that all vertices are consistent, then all conditions of Lemma 15 have been satisfied and hence $G$ is a minor of $H$.

The complexity of the algorithm depends on the time needed to resolve each consistency and the number of resolutions needed. To determine the latter number, we observe that each resolution results in at least one end of an interval being moved. As each end can be moved at most $n$ times and there are in all at most $2n$ ends, there cannot be more than $O(n^2)$ resolutions in all.

In order to resolve an inconsistency, we must first detect an inconsistency and then find the associated leftmost consistent edge. As there are $O(n)$ edges of $G$, detection can be accomplished in linear time. If there is an inconsistency involving $u = r(f(a))$ and $v = r(f(b))$ in $H$, then the leftmost consistent edge can be found in $O(n)$ time by finding the first vertex $u'$ to the right of $u$ on track track$(u)$ such that $u'$ has at least one edge to a vertex on track track$(v)$. We choose $v'$ to be the neighbor of $u'$ such that track$(v')$ = track$(v)$ and $v'$ is the first such neighbor to the right of $v$, if such a vertex exists. If not, we slide $v$ to the right; since there are no transpositions, we will either find an appropriate edge or "fall off" the track in time $O(n)$. Finally, we must reassign endpoints of intervals, which can again be accomplished in $O(n)$ time, for a time bound of $O(n^3)$ for the algorithm. □

When trying to extend the algorithm to $k$-paths, we encounter the difficulty that a star of cross edges (a set of cross edges with one common endpoint) can map into a hair of $H$. Again, if $H$ is a full $k$-path and $G$ a partial $k$-path, there is no need to use a hair of $H$ in this fashion, and the minor containment algorithm extends in the same fashion as the topological embedding algorithm in Theorem 3.

**Theorem 6** *For $G$ a $k$-connected partial $k$-path and $H$ a full $k$-path, it is possible to determine whether or not $G$ is a minor of $H$ in $O(n^3)$ time.* □

`Hole 11:  add proof`

# 6 Subgraph isomorphism algorithm

Due to the more constrained nature of subgraph isomorphism, we are able to prove structural results that result in an $n^{O(1)}$ algorithm for $G$ and $H$ both $k$-connected partial $k$-paths. In the remainder of this section we assume that $G$ and $H$ are $k$-connected partial $k$-paths. For the purposes of illustration, Figure 6 illustrates an embedding of $G$ (Figure 2.1) in $H$ (Figure 2.1).

**Lemma 18** *For $(f, \phi)$ a subgraph isomorphism embedding of $G$ into $H$ and $(P, \chi, hair)$ any compressed path decomposition of $H$, $|V(H)| \geq k+2$,* `Hole A3:  check this constraint` *the following conditions must hold:*

1. *for each end $E$ of $G$, there must exist $p_r$, $r$ even, such that $\chi(p_r)$ contains images of each vertex in $E$; and*

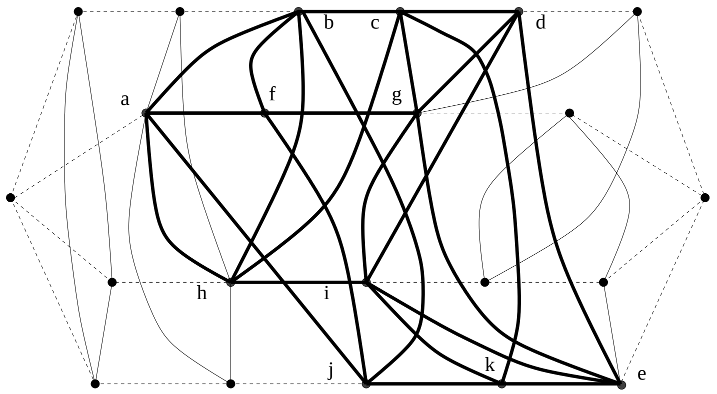


Figure 7: A subgraph isomorphism embedding of $G$ in $H$

2. *for a the initial or final $k$-leaf with neighbourhood $E$, $f(a)$ is either a hair with attachment clique $\chi(p_r)$ or $f(a)$ is either exit($p_{r-1}$) or entry($p_{r+1}$).*

**Proof:** The proof of the first statement, as in the proof of Lemma 7, follows from Lemma 2, point 1.

We prove the second statement by contradiction, considering all the possible locations of $f(a)$. Clearly if $f(a)$ is a hair, it has exactly $k$ neighbours; for each edge involving $a$ to map to an edge in $H$, it must have attachment clique $\chi(p_r)$, as claimed.

By the construction of a compressed path decomposition, since $\text{entry}(p_i) \neq \text{exit}(p_i)$ for each $p_i$ in $P$, in particular $p_{r-2}$ and $p_{r+2}$ each lack one node in the image of $E$. In order for all edges between $a$ and $E$ to exist, either $a = \text{exit}(p_{r-1})$, or $a = \text{entry}(p_{r+1})$, as claimed. $\square$

**Lemma 19** *For $G$ and $H$ $k$-connected partial $k$-paths, there is a subgraph of $H$ isomorphic to $G$ if and only if for any compressed path decomposition of $H$ there exists a subgraph isomorphism embedding $(f, \phi)$, and track layouts $(h_G, \pi_G)$ of $G$ and $(h_H, \pi_H)$ of $H$ such that for $a_I$ and $a_F$ the initial and final $k$-leaves of $G$:*

1. *for all $a \in V(G) \setminus \{a_I, a_F\}$, track($a$) = track($f(a)$);*

2. *for vertices $a$ and $b$ in $V(G)$ such that track($a$) = track($b$) and position($a$) = $k$ + position($b$), position($f(a)$) = $k$ + position($f(b)$);*

3. *for $\chi(p_r)$ containing the images of each vertex in the head, $r$ even, $f(a_I)$ is either a hair with attachment clique $\chi(p_r)$ or exit($p_{r-1}$); and*

4. *for $\chi(p_r)$ containing the images of each vertex in the tail, $r$ even, $f(a_F)$ is either a hair with attachment clique $\chi(p_r)$ or entry($p_{r-1}$).*

**Proof:** Suppose $(f, \phi)$ is a subgraph isomorphism embedding of $G$ into $H$, $(P, \chi, \text{hair})$ a compressed path decomposition of $H$ and $(P_G, \chi_G)$ the path decomposition of $G$ induced by $(f, \phi)$. We can clearly fix a track layout of $H$ based on $(P, \chi)$ and create a layout for $G$ dictated by the tracks in $H$ on which each vertex of $G$ is found, satisfying condition 1. Condition 2 is then a consequence of the fact that the subgraph isomorphism embedding must map adjacent track vertices in $G$ to adjacent track vertices in $H$.

The proofs of conditions 3 and 4 are similar; we consider condition 3 below. By Lemma 18, we know that there exists a vertex $p_r$ in $H$ containing all images of the head, and that $f(a_I)$ is either a hair with attachment clique $\chi(p_r)$, $\text{exit}(p_{r-1})$, or $\text{entry}(p_{r+1})$.

To complete the proof it will suffice to show that $f(a) \neq \mathrm{entry}(p_{r+1})$. Since the head of $G$ separates $a_I$ from the rest of $G$, $\chi(p_r)$ will separate $f(a_I)$ from the images of the rest of $G$. By conditions 1 and 2, $\mathrm{entry}(p_{r+1})$ will be the image of a vertex on track $\mathrm{track}(\mathrm{entry}(p_{r+1}))$, and hence cannot be the image of $a_I$. □

Our algorithm proceeds by fixing a path decomposition and total order on $H$ and then for each track layout of $G$ mapping the head vertices to a size-$k$ bag in $H$. For such a mapping, the initial $k$-leaf is a hair or the track predecessor of the image of a head vertex. The body vertices are mapped as follows: the entry vertex of the next bag is mapped to the next vertex on the appropriate track of $G$. This procedure is repeated until all but the final $k$-leaf is mapped. The final $k$-leaf is mapped to a hair or the track successor of the image of a tail vertex. Finally, hairs and edges are checked to determine if the mapping yields a subgraph isomorphism embedding.

```
form a compressed path decomposition (P, χ, hair) of H
extract a track layout and total order of H
for each track layout of G
   for each size-k bag χ(p_i) in H
      for all t, set f(a_t) = u_t for a_t the track t vertex in
         the head of G and u_t the track t vertex in χ(p)
      for all t, set the current position current(t) to be 1
      set the current position c in H to be i + 1
      ** map initial k-leaf **
      if |hair(p_i)| > 1
         set f(a) to a member of hair(p_i),
            where a is the initial k-leaf of G
      else
         for each track predecessor b of a vertex in p_i, set f(a) = b and
            check for edges from f(a) to each f(a_t)
            (if no such vertex or edges, stop and declare failure)
      ** map body vertices **
      while there are unassigned body vertices in G
         set t to be track(entry(p_c))
         for a the node of G at position current(t) + 1 on track t
         (if no such node, stop and declare failure)
            set f(a) = entry(p_c)
            set current(t) = current(t) + 1
         set c = c + 2
         (if p_c does not exist in P, stop and declare failure)
      ** map final k-leaf **
      if |hair(p_{c-1})| > 1
         set f(a) to a member of hair(p_{c-1}),
            where a is the final k-leaf of G
      else
         for each track successor b of a vertex f(b_t) where b_t is
            the last vertex mapped to track t, set f(a) = b and
            check for edges from f(a) to each f(a_t)
            (if no such vertex or edges, stop and declare failure)
      ** map cross edges **
      check all cross edges (a, b) to see if there is an edge f(a, b)
```

```
(if no such edge, stop and declare failure)
** map hairs **
for each attachment clique in G with s hairs, determine if s ≤ |hair(p)| for p
the vertex in P containing the preimages of attachment clique vertices
(if not, stop and declare failure)
```

**Theorem 7** *For $G$ and $H$ $k$-connected partial $k$-paths, it is possible to determine whether or not $G$ is isomorphic to a subgraph of $H$ in $O(n^3)$ time.* □

**Proof:** Our algorithm compares all layouts of $G$ to a fixed layout of $H$, trying all size-$k$ bags of $H$ as images of the head of $G$ (justified by Lemma 18). We then map each track in $G$ to the appropriate track in $H$ and try possible mappings of the initial and final $k$-leafs, as dictated by Lemma 19. The mapping of hairs to hairs follows from Lemma 19; since body vertices in $G$ map to contiguous track vertices in $H$, the only remaining vertices in $H$ with the appropriate attachment cliques are hairs.

The complexity follows from the facts that there are $O(1)$ track layouts of $G$, $O(n)$ choices of size-$k$ bags in $(P, \chi, \mathrm{hair})$, and for each choice, a linear number of vertices to map and edges to check. □

# 7 Algorithms for the general case

By using dynamic programming, we can solve the topological embedding and minor containment problems for partial $k$-paths (not necessarily proper), though the running times are $n^{k+O(1)}$ rather than $n^{O(1)}$. The dynamic programming subproblem essentially checks if an "initial segment" of $G$ maps into an "initial segment" of $H$. As in our earlier algorithms, we fix a track layout and total ordering of body vertices of $H$, and try all $2(k!)$ possible track layouts of $G$. For a vertex $x$ of $H$, we define $\mathrm{Hpred}(x)$ to be the predecessor of $x$ in the total ordering, and $\mathrm{rank}(x)$ to be the rank of $x$ in the total ordering. For each layout of $G$, we compute all of the entries in a table $T$ defined by a recurrence relation.

In order to focus on specific types of embeddings, similar to those described in Lemmas 8 and 14 for $k$-connected partial proper $k$-paths, we need to prove analogous lemmas for general $k$-connected partial $k$-paths.

**Lemma 20** *For $G$ and $H$ $k$-connected partial $k$-paths, $G$ is topologically embeddable in $H$ if and only if there exists a topological embedding $(f, \phi)$ and track layouts $(h_G, \pi_G)$ of $G$ and $(h_H, \pi_H)$ of $H$ such that for $a_I$ and $a_F$ the initial and final $k$-leaves of $G$ and $u_I$ and $u_F$ the initial and final $k$-leaves of $H$, all conditions except condition 3 in Lemma 8 hold, as well as the following additional conditions:*

*T1. for $a$ a hair in $G$, $f(a)$ is a hair in $H$; and*

*T2. for each cross edge $(a, b)$ in $G$, $\phi((a, b))$ consists of either edge $(f(a), f(b))$ or the path of length two from $f(a)$ to $f(b)$ through a hair.*

**Proof:** As in the proof of Lemma 8, we suppose $(f, \phi)$ is a topological embedding of $G$ into $H$, $(P, \chi)$ a path decomposition of $H$ and $(P_G, \chi_G)$ the path decomposition of $G$ induced by $(f, \phi)$. We fix a track layout of $H$ and create a layout for $G$ dictated by the tracks in $H$ on which each vertex of $G$ is found.

We mimic the proof of Lemma 8 to show that each condition must hold (as in that lemma, the proof of the reverse implication is obvious). The proof of condition 4 is identical to that in Lemma 8 and hence omitted. We prove the remaining conditions by supposing instead that one was violated, and finding the first location in the total order of $H$ at which a violation occurs.

The cases for condition 1 or 2 being violated first are identical to those in Lemma 8. Since the proof for condition T1 is the same as of condition 3, it suffice to consider the remaining case below.

Suppose that condition T1 is violated first, namely, that a hair $a$ maps to a vertex $f(a)$ on track $t$ for $t > 0$ and that $\chi(p_i)$ is the bag in $P$ in which $f(a)$ appears for the first time; by condition 2 of Lemma 2, we know that each of $\chi(p_{i-1})$ and $\chi(p_{i-1})$ contain at exactly one vertex from each track. In particular, $f(a)$ is the vertex from track $t$ in $\chi(p_{i-1})$. Since $a$ is not on track $t$, the corresponding bag in $(P_G, \chi_G)$ violates condition 2 of Lemma 2. This completes the proof of the lemma. □

**Lemma 21** *For $G$ and $H$ $k$-connected partial $k$-paths, $G$ is a minor of $H$ if and only if there exist a minor embedding $(f, \xi)$ and track layouts $(h_G, \pi_G)$ and $(h_H, \pi_H)$ such that, for $a_I$ and $a_F$ the initial and final $k$-leaves of $(h_G, \pi_G)$ and $u_I$ and $u_F$ the initial and final $k$-leaves of $(h_H, \pi_H)$, all conditions except condition 1 in Lemma 14 hold, as well as the following additional conditions:*

*M1. for $a$ a hair in $G$, $f(a)$ is a hair in $H$; and*

*M2. for all $a \in V(G) \setminus \{a_I, a_F\}$, $f(a)$ is an interval of vertices $[\ell(f(a)), r(f(a))] = [u, v]$ on the track of $a$ and zero or more hairs in $H$ where each hair has a distinct attachment clique.*

**Proof:** As the lemma is very similar to Lemma 14, we present only the necessary modifications to that proof. The proof of condition 4 of Lemma 14 holds for partial $k$-paths. Furthermore, we first observe that the proof of condition M1 follows from the proof of condition T1 of Lemma 20. Since conditions 2 and 4 of Lemma 14 follow from condition M2, it suffices to prove this condition.

We can satisfy condition M2 for the neighbours of $a \in \{a_I, a_F\}$, using the same argument as in the proof of Lemma 14. As in the argument concerning condition 1 of Lemma 14, we can use a proof by induction to show that no track other than track$(a)$ can be used to map $a$.

To complete the proof, we must show that it is possible to include in $f(a)$ at most one hair with a particular attachment clique. Suppose instead $a$ included two hairs $h_1$ and $h_2$ with attachment clique $A$, where an edge $(a, b)$ was mapped to $(h_1, u_b)$, for $u_b$ a node in $f(b)$, and an edge $(a, c)$ was mapped to $(h_2, u_c)$, for $u_c$ a node in $f(c)$. Since $h_1$ and $h_2$ have the same attachment clique, the existence of the edge $(h_2, u_c)$ implies the existence of the edge $(h_1, u_c)$. We can then alter the mapping so that each $(a, b)$ and $(a, c)$ are mapped to $(h_1, u_b)$ and $(h_1, u_c)$. By repeating this process, we obtain our result. □

First, we define some useful notation. For any vertex $x$ in $H$, let $U(x)$ be the set of vertices of $H$, one per track, that are the rightmost vertices preceding or equal to $x$ (that is, $U(x) \cup \{\text{pred}(x)\}$ is the bag of the compressed path decomposition of $H$ in which $x$ appears for the first time). Let $H(x)$ be the subgraph of $H$ induced by all vertices of rank less than or equal to rank$(x)$ and every hair whose attachment clique contains only vertices of rank less than or equal to rank$(x)$. We call a set of vertices in $G$ (or in $H$) a *spanning set* if it contains exactly one vertex per track. For a spanning set $A$ of $G$, let $G(A)$ be the graph induced by all vertices in $A$, all vertices to the left of vertices in $A$, and all hairs whose attachment clique contains only vertices in $A$ or to the left of vertices in $A$. Note that since there is a total ordering on the body vertices of $H$, $H(v) = H(U(v))$ for any vertex $v$.

The topological embedding algorithm computes all quantities $T[A, x, \alpha]$, where $A$ is a spanning set of $G$, $x$ is a vertex in $H$ with rank$(x) \geq k$, and $\alpha$ is a subset of $\{1, 2, \ldots, k\}$ representing a subset of tracks. The entry $T[A, x, \alpha]$ has value "true" if and only if $G(A)$ can be topologically embedded into $H(x)$, where the topological embedding respects tracks (fulfils all but condition 4 of Lemma 20) and maps a vertex $a$ in $A$ to the vertex in $U(x)$ on track track$(a)$ if and only if track$(a) \in \alpha$. For $A$ a spanning set of $G$ or $H$, we define $h(A)$ to be the number of hairs with attachment clique $A$. Hairs in $H$ can be used to embed hairs in $G$ or cross edges (one per hair).

Clearly there are $O(n^k)$ choices for $A$, $O(n)$ choices for $x$, and $2^k$ choices for $\alpha$, so if an entry in $T$ can be computed in time $n^{O(1)}$, this will result in an $n^{k+O(1)}$ algorithm. We now detail the recurrence relation for $T$.

The base case consists of entries $T[A_{\text{head}}, x, \alpha]$, where $A_{\text{head}}$ is the set of vertices in the head of $G$, and $x$ is any vertex in $H$ with $\text{rank}(x) \geq k$. There are two cases. If $\text{rank}(x) = k$, $T[A_{\text{head}}, x, \alpha]$ is true if and only if:

A1. $\alpha = \{1, 2, \ldots, k\}$;

A2. $h(U(x)) \geq h(A_{\text{head}})$;

A3. for all but $h(U(x)) - h(A_{\text{head}})$ cross edges between vertices $b, c$ in $A_{\text{head}}$, there is an edge between the vertices in $U(x)$ on tracks $\text{track}(b)$ and $\text{track}(c)$.

If $\text{rank}(x) > k$, $T[A_{\text{head}}, x, \alpha]$ is true if and only if:

B1. $\alpha = \{1, 2, \ldots, k\}$;

B2. $h(U(x)) \geq h(A_{\text{head}}) - 1$;

B3. for all but $h(U(x)) - h(A_{\text{head}}) + 1$ cross edges between vertices $b, c$ in $A_{\text{head}}$, there is an edge between the vertices in $U(x)$ on tracks $\text{track}(b)$ and $\text{track}(c)$.

For the recurrence, given an entry $T[A, x, \alpha]$, let $a$ be the vertex in $G$ with $\text{track}(a) = \text{track}(x)$. There are two cases, depending on whether or not $\text{track}(a) \in \alpha$.

If $\text{track}(a) \in \alpha$, then $T[A, x, \alpha]$ is true if and only if the following conditions are true:

C1. there are no cross edges from $a$ to vertices to the left of vertices in $A$;

C2. either $T[A', \text{pred}(x), \alpha \setminus \{\text{track}(a)\}]$ is true or $T[A', \text{Hpred}(x), \alpha]$ is true, where $A' = (A \setminus \{a\}) \cup \{\text{pred}(a)\}$;

C3. if $h(A) > 0$, then $|\alpha| = k$;

C4. $h(U(x)) \geq h(A)$;

C5. for all but $h(U(x)) - h(A)$ cross edges incident with $a$ and vertices $b$ in $A$ such that $\text{track}(b) \in \alpha$, there is an edge between $x$ and the vertex in $U(x)$ on track $\text{track}(b)$.

If $\text{track}(a) \notin \alpha$, then $T[A, x, \alpha]$ is true if and only if the following condition is true:

D1. either $T[A, \text{Hpred}(x), \alpha]$ is true, or $T[A, \text{Hpred}(x), \alpha \cup \{\text{track}(a)\}]$ is true.

**Lemma 22** *If $T[A, x, \alpha]$ is true, then $G(A)$ can be topologically embedded into $H(x)$, where the topological embedding respects tracks (fulfils all but condition 4 of Lemma 20) and maps a vertex $b$ in $A$ to the vertex in $U(x)$ on track track($b$) if and only if track($b$) $\in \alpha$.*

**Proof:** We prove the lemma by induction on $\text{rank}(x)$. For the base case, $\text{rank}(x) = k$. Then $U(x)$ is the head of $H$, $A$ must be $A_{\text{head}}$ (otherwise the recurrence would be used to compute $T$, and $\text{pred}(a)$ is not defined) and by condition A1, $\alpha = \{1, 2, \ldots, k\}$. We define the topological embedding of $G(A_{\text{head}})$ into $H(x)$ by mapping $b$ in $A_{\text{head}}$ to the vertex on track $\text{track}(b)$ in $U(x)$. Hairs in $G$ with attachment clique $A_{\text{head}}$ map to hairs in $H$ with attachment clique $U(x)$ (including the initial vertex of $G$, which maps to the initial vertex of $H$). Cross edges in $A_{\text{head}}$ map to either cross edges in

$H$ or paths of length 2 in $H$ going through hairs with attachment clique $U(x)$. Conditions A2 and A3 in the definition of the base case for $T$ ensure that there are enough hairs.

Now assume the statement of the lemma is true for all vertices $x$ of rank less than $n$, for some $n > k$. Suppose $T[A, x, \alpha]$ is true, where $\mathrm{rank}(x) = n$. There are three cases.

If $A = A_{\mathrm{head}}$, then $\alpha = \{1, 2, \ldots, k\}$ by condition B1. We define the topological embedding of $G(A_{\mathrm{head}})$ into $H(x)$ by mapping $b$ in $A_{\mathrm{head}}$ to the vertex on track $\mathrm{track}(b)$ in $U(x)$. Hairs in $G$ with attachment clique $A_{\mathrm{head}}$ map to hairs in $H$ with attachment clique $U(x)$, except the initial vertex of $G$, which maps to the initial vertex of $H$. Edges incident with hairs in $G$ map to edges incident with the corresponding hairs in $H$, except for the edges incident with the initial vertex of $G$, which map to the path from the initial vertex of $H$ to the corresponding vertex of $U(x)$, along the appropriate track. Cross edges in $A_{\mathrm{head}}$ map to either cross edges in $H$ or paths of length two in $H$ going through hairs with attachment clique $U(x)$. Conditions A2 and A3 in the definition of the base case for $T$ ensure that there are enough hairs. The track edges from the initial vertex of $G$ to vertices in $A_{\mathrm{head}}$ map to paths from the initial vertex of $H$ to vertices in $U(x)$.

For the remaining two cases, we assume $A \neq A_{\mathrm{head}}$. If $\mathrm{track}(x) \in \alpha$, then either $T[A', \mathrm{Hpred}(x), \alpha \setminus \{\mathrm{track}(a)\}]$ is true or $T[A', \mathrm{Hpred}(x), \alpha]$ is true (condition C2). By the inductive hypothesis there exists a topological embedding $(f, \phi)$ of $G(A')$ into $H(\mathrm{Hpred}(x))$. We can extend this mapping by having $f$ map $a$ to $x$, and $\phi$ map the track edge $(\mathrm{pred}(a), a)$ to the path from $f(\mathrm{pred}(a))$ to $x$ along track $\mathrm{track}(a)$. The extended $\phi$ will map cross edges $(a, b)$ (where $b \in A$ and $\mathrm{track}(b) \in \alpha$) to either $(f(a), f(b))$, if this edge exists, or to a path of length 2 through a hair of $H$ attached at $A$. Conditions C3, C4, and C5 in the recurrence for $T$ ensure that there are enough hairs in $H$ to embed all such cross edges of $G$ incident on $a$ (which, by condition C1 must have endpoints in $A$) and all hairs in $G$ with attachment clique $A$. This gives a topological embedding of $G(A)$ into $H(x)$ meeting the conditions in the statement of the lemma.

If $\mathrm{track}(x) \notin \alpha$, then by condition D1, either $T[A, \mathrm{Hpred}(x), \alpha]$ is true, or $T[A, \mathrm{Hpred}(x), \alpha \cup \{\mathrm{track}(a)\}]$ is true. In either case, by the inductive hypothesis, there exists a topological embedding $(f, \phi)$ of $G(A)$ into $H(\mathrm{Hpred}(x))$. This topological embedding meets the conditions in the statement of the lemma. □

**Lemma 23** *If $G(A)$ can be topologically embedded into $H(x)$, where the topological embedding respects tracks (fulfils all but condition 4 of Lemma 20) and maps a vertex $b$ in $A$ to the vertex in $U(x)$ on track $\mathrm{track}(b)$ if and only if $\mathrm{track}(b) \in \alpha$, then $T[A, x, \alpha]$ is true.*

**Proof:**

The proof proceeds by induction on $\mathrm{rank}(x)$. For the base case, $\mathrm{rank}(x) = k$, and $U(x)$ is the head of $H$. Since there are exactly $k$ body vertices in $G(A)$, $A = A_{\mathrm{head}}$. Suppose that there is a topological embedding $(f, \phi)$ from $G(A_{\mathrm{head}})$ into $H(x)$ as described in the statement of the lemma. Because the vertices in $U(x)$ have no track predecessors, $f$ must map a vertex $a$ in $A_{\mathrm{head}}$ to the vertex of $U(x)$ on track $\mathrm{track}(a)$, and $\alpha$ must equal $\{1, 2, \ldots, k\}$. A hair in $G$ with attachment clique $A_{\mathrm{head}}$ (including the initial vertex of $G$) must map into a hair of $U(x)$, and cross edges between vertices in $A_{\mathrm{head}}$ must map either to edges between vertices in $U(x)$ or paths of length 2 through hairs attached at $U(x)$. Thus conditions A1–A3 are satisfied, and $T[A_{\mathrm{head}}, x, \alpha]$ is true.

Now assume the statement of the lemma is true for all vertices $x$ of rank less than $n$, for some $n > k$. Suppose $\mathrm{rank}(x) = n$, and there is a topological embedding $(f, \phi)$ from $G(A)$ into $H(x)$ as described in the statement of the lemma.

If $A = A_{\mathrm{head}}$, we can conclude as before that $f$ must map a vertex $a$ in $A_{\mathrm{head}}$ to the vertex of $U(x)$ on track $\mathrm{track}(a)$, and $\alpha$ must equal $\{1, 2, \ldots, k\}$. The initial vertex of $G$ must map into the initial vertex of $H$, and cross edges between vertices in $A_{\mathrm{head}}$ must map either to edges between vertices in $U(x)$ or paths of length 2 through hairs attached at $U(x)$. Thus conditions B1–B3 are satisfied, and $T[A_{\text{head}}, x, \alpha]$ is true.

Assuming $A \neq A_{\text{head}}$, let $a$ be the vertex in $A$ on track track$(x)$, and $A'$ be $A \setminus \{a\} \cup \{\text{pred}(a)\}$. There are two cases, depending on whether or not $f(a) = x$.

If $f(a) \neq x$, then $f$ maps $a$ to a track predecessor of $x$ and $f$ topologically embeds $G(A)$ into $H(\text{Hpred}(x))$. If $f(a) = \text{pred}(x)$, then by the inductive hypothesis, $T[A', \text{Hpred}(x), \alpha \cup \{\text{track}(a)\}]$ is true. If $f(a) \neq \text{pred}(x)$, then by the inductive hypothesis, $T[A', \text{Hpred}(x), \alpha\}]$ is true. Thus condition D1 is satisfied, and $T[A, x, \alpha]$ is true.

If $f(a) = x$ and $f(\text{pred}(a))$ is to the left of $\text{pred}(x)$, then $f$ restricted to $G(A')$ topologically embeds $G(A')$ into $H(\text{Hpred}(x))$, and by the inductive hypothesis, $T[A', \text{Hpred}(x), \alpha \setminus \{\text{track}(a)\}]$ is true. If $f(a) = x$ and $f(\text{pred}(a)) = \text{pred}(x)$, then $f$ restricted to $G(A')$ topologically embeds $G(A')$ into $H(\text{Hpred}(x))$, and by the inductive hypothesis $T[A', \text{Hpred}(x), \alpha]$ is true. This proves that condition C2 holds. To see that condition C1 holds, suppose there were a cross edge $(a, b)$ with $b$ to the left of some vertex $c$ in $A$. This must map onto either a cross edge $(f(a), f(b))$ in $H(x)$, or onto a path of length 2 going through a hair of $H(x)$. Either of these imply that $f(a)$ and $f(b)$ are in a bag together, but there is only one bag in $H(x)$ containing $f(a) = x$, namely $U(x)$, and $U(x)$ does not contain $f(b)$ since it is to the left of $f(c)$. Condition C3 holds because a hair attached at $A$ must map onto a hair attached at $U(x)$ (since $x$ does not appear in any other bag). Finally, conditions C3 and C4 follow from seeing how cross edges incident with $a$ and hairs attached at $A$ map into $H(x)$. So $T[A, x, \alpha]$ is true. □

**Theorem 8** *For $G$ and $H$ $k$-connected partial $k$-paths, $G$ can be topologically embedded in $H$ if and only if there exist track layouts of $G$ and $H$ and a vertex $v$ in $H$ such that $T[A_{tail}, v, \{1, 2, \ldots, k\}]$ is true, where $A_{tail}$ is the set of vertices in the tail of $G$.*

**Proof**: Suppose $G$ can be topologically embedded in $H$. Fix a track layout of $G$, and the corresponding track layout of $H$. Let $v$ be the vertex of highest rank in $H$ which is the image of some vertex in $G$. Then $G = G(A_{\text{tail}})$ can be topologically embedded into $H(v)$, and so $T(A_{\text{tail}}, v, \{1, 2, \ldots, k\}]$ is true.

If $T[A_{\text{tail}}, v, \{1, 2, \ldots, k\}]$ is true, then $G(A_{\text{tail}}) = G$ can be topologically embedded into $H(v)$, which is a subgraph of $H$. □

**Theorem 9** *For $G$ and $H$ $k$-connected partial $k$-paths, it is possible to determine in $n^{k+O(1)}$ time whether or not $G$ is topologically embeddable in $H$.*

**Proof:** There are $n^{k+O(1)}$ entries in the table $T$, and if the hair function $h$ is precomputed (which can be done in $O(n)$ time), each entry in $T$ can be computed in constant time. □

A similar idea will work for the minor embedding problem; the key differences are due to the fact that in a minor embedding, a vertex of $G$ maps to a track segment of $H$, and a hair of $H$ can be used to embed a star of cross edges in $G$. For this problem, the quantities computed are $M[A, x, \beta]$, where $A$ is a spanning set of $G$, $x$ a vertex in $H$ with $\text{rank}(x) \geq k$, and $\beta$ is a subset of $\Omega = \{\{i, j\} | 1 \leq i, j \leq k, i \neq j\}$ representing track numbers of endpoints of cross edges. Given such a $\beta$, let $G_\beta(A)$ be the graph with vertex set $V(G(A))$ and edge set $E(G(A)) \setminus \{(b, c) | b, c \in A, \{\text{track}(b), \text{track}(c)\} \notin \beta\}$. That is, the sets of pairs representing track numbers of endpoints of cross edges in $G_\beta(A)$ is a subset of $\beta$. Given a spanning set $A$ in $G$ (or $H$), let star$(A, i)$ be the set of cross edges in $G$ (or $H$ respectively) with both endpoints in $A$ and one endpoint on track $i$.

The entry $M[A, x, \beta]$ has value "true" if and only if there is a minor embedding from $G_\beta(A)$ to $H(x)$, where the minor embedding respects tracks (satisfies all but condition 4 of Lemma 21).

There are $O(n^k)$ choices for $A$, $O(n)$ choices for $x$, and $2^{k(k-1)}$ choices for $\beta$, so an $n^{k+O(1)}$ algorithm will result if each table entry can be computed in $n^{O(1)}$ time.

The base case for the recurrence relation for $M$ consists of entries $M[A_{\text{head}}, x, \beta]$, where $\text{rank}(x) = k$. In this case, $M[A_{\text{head}}, x, \beta]$ is true if and only if:

E1. $h(U(X)) \geq h(A_{\text{head}})$;

E2. There is a set $S \subseteq \{1, 2, \ldots, k\}, |S| \leq h(U(X)) - h(A_{\text{head}})$, such that for every cross edge $(b, c)$ in $G_\beta(A_{\text{head}})$ such that $b, c \in A_{\text{head}}$ and $\{\text{track}(b), \text{track}(c)\} \in \beta$, either $(b, c) \in \text{star}(A_{\text{head}}, i)$ for some $i \in S$, or there is an edge between the vertices in $U(x)$ on tracks $\text{track}(b)$ and $\text{track}(c)$.

For the recurrence, given an entry $M[A, x, \beta]$, let $a$ be the vertex in $G$ with $\text{track}(a) = \text{track}(x)$. $M[A, x, \beta]$ is true if and only if one of the following two conditions is true:

F1. $M[A', \text{Hpred}(x), \Omega]$ is true, where $A' = (A \setminus \{a\}) \cup \{\text{pred}(a)\}$, and there is a set $S \subseteq \{1, 2, \ldots, k\}$, $|S| \leq h(U(X)) - h(A)$, such that for every cross edge $(b, c)$ in $G_\beta(A)$ such that $b, c \in A$ and $\{\text{track}(b), \text{track}(c)\} \in \beta$, either $(b, c) \in \text{star}(A, i)$ for some $i \in S$, or there is an edge between the vertices in $U(x)$ on tracks $\text{track}(b)$ and $\text{track}(c)$. (Note that this requires that $A \neq A_{\text{head}}$.)

F2. There are sets $\gamma \subseteq \beta$ and $S \subseteq \{1, 2, \ldots, k\}, |S| \leq h(U(X)) - h(A)$, such that $M[A, \text{Hpred}(x), \gamma]$ is true, and for every cross edge $(b, c)$ in $G_\beta(A)$ such that $b, c \in A$ and $\{\text{track}(b), \text{track}(c)\} \in \beta \setminus \gamma$, either $(b, c) \in \text{star}(A, i)$ for some $i \in S$, or there is an edge between the vertices in $U(x)$ on tracks $\text{track}(b)$ and $\text{track}(c)$.

Intuitively, condition F1 corresponds to the case where the image of $a$ in $H$ begins at the vertex $x$, and condition F2 corresponds to the case where the image of $a$ is stretched to include the vertex $x$.

**Lemma 24** *If $M[A, x, \beta]$ is true, then there is a minor embedding of $G_\beta(A)$ into $H(x)$, where the minor embedding respects tracks (fulfils all but conditions 4 of Lemma 21).*

**Proof:** We prove the lemma by induction on $\text{rank}(x)$. For the base case, $\text{rank}(x) = k$. Then $U(x)$ is the head of $H$ and $A$ must be $A_{\text{head}}$ (otherwise the recurrence would be used to compute $M$, and $\text{pred}(a)$ is not defined). We define the minor embedding $(f, \xi)$ of $G_\beta(A_{\text{head}})$ into $H(x)$ as follows. For a vertex $b$ in $A_{\text{head}}$, if $\text{track}(b) \notin S$, $f(b)$ is the vertex in $U(x)$ on track $\text{track}(b)$. Otherwise $f(b)$ consists of this vertex and one of the hairs with attachment clique $U(x)$, where the mappings are chosen so that each such hair is in at most one $f(b)$. Hairs in $G$ with attachment clique $A_{\text{head}}$ map to hairs in $H$ with attachment clique $U(x)$ (including the initial vertex of $G$, which maps to the initial vertex of $H$). Conditions E1 and E2 ensure that there are enough hairs to define $f$ this way. Edges incident with hairs in $G$ map to edges incident with the corresponding hairs in $H$. Given a cross edge $(b, c)$, $b, c \in A_{\text{head}}$, $\{\text{track}(b), \text{track}(c)\} \in \beta$, if either $|f(b)| > 1$ or $|f(c)| > 1$ (without loss of generality $|f(b)| > 1$, $\text{track}(b) = i$ for some $i \in S$) then $\xi$ maps $(b, c)$ to the edge $(v, w)$, where $v$ is the hair in $f(b)$ and $w$ is the vertex in $U(x)$ on track $\text{track}(c)$; otherwise $(b, c)$ maps to $(f(b), f(c))$. Condition E2 ensures that the resulting $(f, \xi)$ satisfies the definition of minor embedding.

Now assume the statement of the lemma is true for all vertices $x$ of rank less than $n$, for some $n > k$. Suppose $M[A, x, \beta]$ is true, where $\text{rank}(x) = n$. There are two cases. Either condition F1 is true, or condition F2 is true.

In the first case, condition F1 is true, $M[A', \text{Hpred}(x), \Omega]$ is true, and by the inductive hypothesis there exists a minor embedding $(f, \xi)$ of $G_\Omega(A') = G(A')$ into $H(\text{Hpred}(x))$. We extend this to a minor embedding of $G_\beta(A)$ into $H(x)$ as follows. First, we define $f(a) = \{x\}$. Then for a vertex $b$ in $A$, if $\text{track}(b) \in S$, we add to $f(b)$ one of the hairs with attachment clique $U(x)$, where the mappings are chosen so that each such hair is in at most one $f(b)$. Hairs in $G$ with attachment clique $A$ map to hairs in $H$ with attachment clique $U(x)$. We extend $\xi$ as follows: edges incident with hairs in $G$ map to edges incident with the corresponding hairs in $H$. Given a cross edge $(b,c)$, $b,c \in A$, $\{\text{track}(b), \text{track}(c)\} \in \beta$, if either $|f(b)| > 1$ or $|f(c)| > 1$ then without loss of generality $|f(b)| > 1$, $\text{track}(b) = i$ for some $i \in S$, and $\xi$ maps $(b,c)$ to the edge $(v,w)$, where $v$ is the hair in $f(b)$ and $w$ is the vertex in $U(x)$ on track $\text{track}(c)$; otherwise $(b,c)$ maps to $(f(b), f(c))$. Condition F1 ensures that the resulting $(f,\xi)$ satisfies the definition of minor embedding.

In the second case, $M[A, \text{Hpred}(x), \gamma]$ is true, and by the inductive hypothesis there exists a minor embedding $(f,\xi)$ of $G_\gamma(A)$ into $H(\text{Hpred}(x))$. We extend this to a minor embedding of $G_\beta(A)$ into $H(x)$ as follows. For a vertex $b$ in $A$, if $\text{track}(b) \in S$, we add to $f(b)$ one of the hairs with attachment clique $U(x)$, where the mappings are chosen so that each such hair is in at most one $f(b)$. Hairs in $G$ with attachment clique $A$ map to hairs in $H$ with attachment clique $U(x)$. We extend $\xi$ as follows: edges incident with hairs in $G$ map to edges incident with the corresponding hairs in $H$. Given a cross edge $(b,c)$, $b,c \in A$, $\{\text{track}(b), \text{track}(c)\} \in \beta \setminus \gamma$, if either $|f(b)| > 1$ or $|f(c)| > 1$ then without loss of generality $|f(b)| > 1$, $\text{track}(b) = i$ for some $i \in S$, and $\xi$ maps $(b,c)$ to the edge $(v,w)$, where $v$ is the hair in $f(b)$ and $w$ is the vertex in $U(x)$ on track $\text{track}(c)$; otherwise $(b,c)$ maps to $(f(b), f(c))$. Condition F2 ensures that the resulting $(f,\xi)$ satisfies the definition of minor embedding. □

**Lemma 25** *If there is a minor embedding of $G_\beta(A)$ into $H(x)$, where the minor embedding respects tracks (fulfils all but condition 4 of Lemma 21), then $M[A,x,\beta]$ is true.*

**Proof:**

The proof proceeds by induction on $\text{rank}(x)$. For the base case, $\text{rank}(x) = k$, and $U(x)$ is the head of $H$. Since there are exactly $k+1$ body vertices in $G_\beta(A)$, $A = A_{\text{head}}$. Suppose that there is a minor embedding $(f,\phi)$ from $G_\beta(A_{\text{head}})$ into $H(x)$ as described in the statement of the lemma. Because the vertices in $U(x)$ have no track predecessors, $f$ must map a vertex $a$ in $A_{\text{head}}$ to the vertex of $U(x)$ on track $\text{track}(a)$. A hair in $G$ attached at $A_{\text{head}}$ (including the initial vertex of $G$) must map into a hair of $H$ attached at $U(x)$. Let $Y$ be the set of hairs of $H$ attached at $U(x)$ that are not images of hairs in $G$; clearly $|Y| \leq h(U(x) - h(A_{\text{head}})$. Let $S = \{\text{track}(b) | \exists y \in Y \text{ such that } y \in f(b)\}$. Given a cross edge $(b,c)$, $b,c \in A_{\text{head}}$, it must map either to a cross edge between vertices in $U(x)$ or to an edge from a hair in $Y$ to vertices in $U(x)$, and in the latter case $(b,c) \in \text{star}(A_{\text{head}}, i)$ for $i \in Y$. Thus conditions E1 and E2 are satisfied, and $M[A,x,\beta]$ is true.

Now assume the statement of the lemma is true for all vertices $x$ of rank less than $n$, for some $n > k$. Suppose $\text{rank}(x) = n$, and there is a minor embedding $(f,\xi)$ from $G_\beta(A)$ into $H(x)$ as described in the statement of the lemma.

The image $f(a)$ must contain $x$. If it does not contain $\text{Hpred}(x)$, then $f(\text{pred}(a))$ is a track segment (plus some hairs) whose right endpoint is $\text{Hpred}(x)$. Restricting $(f,\xi)$ to $G(A') = G_\Omega(A')$ and $H(\text{Hpred}(x))$ shows that, by the inductive hypothesis, $M[A', \text{Hpred}(x), \Omega]$ is true. As above, let $Y$ be the set of hairs of $H$ attached at $U(x)$ that are not images of hairs in $G$; clearly $|Y| \leq h(U(x) - h(A)$. Let $S = \{\text{track}(b) | \exists y \in Y \text{ such that } y \in f(b)\}$. Given a cross edge $(b,c)$, $b,c \in A$, it must map either to a cross edge between vertices in $U(x)$ or to an edge from a hair in $Y$ to vertices in $U(x)$, and in the latter case $(b,c) \in \text{star}(A,i)$ for $i \in Y$. Thus conditions F1 is satisfied, and $M[A,x,\beta]$ is true.

If $\text{Hpred}(x) \in f(a)$, then let $\delta_1$ be the set of all pairs $\{\text{track}(a), \text{track}(b)\}$ such that $b \in A$, $\{\text{track}(a), \text{track}(b)\} \in \beta$, and $\xi((a,b))$ maps to a cross edge with endpoints in $U(x)$. Let $\delta_2$ be the set of all pairs $\{\text{track}(b), \text{track}(c)\}$ such that $b,c \in A$, $\{\text{track}(a), \text{track}(b)\} \in \beta$, and $\xi((b,c))$ is incident on a hair of $H$. Let $\gamma = \beta \setminus (\delta_1 \cup \delta_2)$; since $\delta_1, \delta_2 \subseteq \beta$, $\delta_1 \cup \delta_2 = \beta \setminus \gamma$. The minor embedding restricted to $G_\gamma(A)$ and $H(\text{Hpred}(x))$ shows that, by the inductive hypothesis, $M[A, \text{Hpred}(x), \gamma]$ is true. Let $S = \{\text{track}(b) | f(b) \text{ contains a hair of } H\}$; clearly $\delta_2 \subseteq S$. Since the minor embedding maps cross edges $(b,c)$ in $G_\beta(A)$, $b,c \in A$, $\{\text{track}(b), \text{track}c\}$ in $\beta \setminus \gamma$ to either cross edges in $H$ ($\{\text{track}(b), \text{track}(c)\} \in \delta_1$) or to edges incident on a hair ($\{\text{track}(b), \text{track}(c)\} \in \delta_2$), condition F2 is satisfied, and $M[A,x,\beta]$ is true. □

**Theorem 10** *For $G$ and $H$ $k$-connected partial $k$-paths, there is a minor embedding of $G$ into $H$ if and only if there exist track layouts of $G$ and $H$ and a vertex $v$ in $H$ such that $M[A_{tail}, v, \Omega]$ is true, where $A_{tail}$ is the set of vertices in the tail of $G$.*

**Proof**: Suppose there is a minor embedding of $G$ into $H$. Fix a track layout of $G$, and the corresponding track layout of $H$. Let $v$ be the vertex of highest rank in $H$ which is in the image of some vertex in $G$. Then $G = G_\Omega(A_{\text{tail}})$ can be topologically embedded into $H(v)$, and so $T(A_{\text{tail}}, v, \Omega]$ is true.

If $M[A_{\text{tail}}, v, \Omega]$ is true, then there is a minor embedding of $G_\Omega(A_{\text{tail}}) = G$ into $H(v)$, which is a subgraph of $H$. □

**Theorem 11** *For $G$ and $H$ $k$-connected partial $k$-paths, it is possible to determine in $n^{k+O(1)}$ time whether or not there is a minor embedding of $G$ into $H$.*

**Proof:** There are $n^{k+O(1)}$ entries in the table $T$, and if the hair function $h$ is precomputed (which can be done in $O(n)$ time), each entry in $T$ can be computed in constant time. □

# 8 Extensions and open problems

We can improve the complexity of our algorithms when there are additional constraints on $G$ and $H$. To specify a particular full $k$-path, a track layout and a total order on the vertices (consistent with the partial order imposed by tracks) are sufficient. The neighbors of each vertex are completely determined by the total order. A full proper $k$-path $G$ can thus be represented as a $k$-character string $S_G$ derived from the track numbers of the entry vertices in a path decomposition of $G$ [PRS98]; an extension of this notation allows us to handle full $k$-paths, as well, by associating with each entry vertex $a$ the number $h(a)$ of hairs sharing its attachment clique.

When $G$ and $H$ are both full proper $k$-paths, we can solve subgraph isomorphism by fixing a string representation of $H$ and then executing string matching between $S_H$ and each of the $2(k!)$ possible string representations of $G$. When $G$ and $H$ are both full $k$-paths, we need to determine a matching such that $a \in V(G)$ matches $u \in V(H)$ if and only if $\text{track}(a) = \text{track}(u)$ and the number of hairs with attachment clique $\text{AC}(a)$ is at most the number of hairs with attachment clique $\text{AC}(u)$. This extension of string matching can be solved in time $O(|V(H)|\sqrt{|V(G)|}\log(|V(G)|))$ [AF91], for a total complexity of $O(n\sqrt{n}\log n)$. It turns out that for topological embedding of full proper $k$-paths, a slight extension of this idea suffices to give an $O(n\sqrt{n}\log n)$ algorithm.

`Hole 14: Add the details.`

The most obvious open problem is to improve the algorithm given in Section 7 to remove of any function of $k$ from the exponent. Beyond that, we suspect that the requirement of $k$-connectivity may yield more useful structural information for partial $k$-trees than has been discovered to date.

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